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A Prehistory of Nim

Lisa Rougetet

Lisa Rougetet (lisa.rougetet@gmail.com) has just completed her Ph.D. in the history of mathematics at Lille 1 University (France) focusing on combinatorial games, from the first arithmetical recreations to the ordered field of surreal numbers. Previously, she received an M.S. in pure mathematics (2009) and an M.A. in history and philosophy of sciences (2011) from the same university. She spent several years with an amateur theater company and has long enjoyed horseback riding, especially dressage.

Bouton's 1901 article on Nim [**3**] is considered the starting point of the relatively recent (i.e., 20th century) mathematical theory of combinatorial games. The game became popular beyond mathematics with the 1961 Alain Resnais film *L'Année dernière à Marienbad* (*Last Year at Marienbad*). It was known under the name *le jeu de Marienbad* (the Marienbad game) and was mainly played by students in cafés and bars to impress friends or win shots of bourbon.

What came before Nim? Imagine a trip back in time to 17th century Europe, at the king's court or in a private curiosity cabinet. Were there games or "recreations" where you can always win if you know the key trick? In this article, we consider several Europe predecessors to Nim.

Bouton's Nim

Nim is easy to play, as long as you have few coins or stones or matches around. A possible starting position is shown in Figure 1: Three rows of stones are set on a table, each row is considered as a pile, and they contain, respectively, 3, 4, and 5 stones. Two players alternate selecting one of the horizontal piles and taking from it as many stones as they want, from one stone up to that whole pile. The player who takes the last stone(s) from the table wins the game. The solution to the game is based on the notion of "safe combinations" [**3**], special positions that a player must reach in order to win the game, no matter how the opponent plays.

To determine whether a position is safe, write the number of stones in each pile in binary. Place these binary numbers in rows so that the binary digits are aligned in vertical columns. If the sum of each column is congruent to 0 mod 2, then the set of numbers on the table forms a safe combination. This carryless addition gives what is called the Nim-sum. In our example, 3, 4, 5 is not a safe combination, so if it is your turn to play, you should remove two stones from the first pile, leaving your opponent the combination 1, 4, 5—verify that this is a safe combination. Can you figure out how to win now, no matter what your opponent does?

Charles Leonard Bouton (1869–1922) both introduced this version of the game and gave its full solution in 1901. He justified his interest "on account of its seeming

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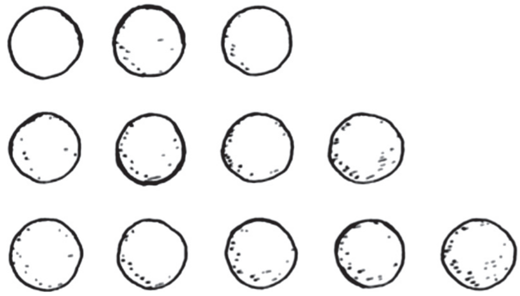

Figure 1. A Nim position: three rows (piles) with 3, 4, and 5 stones [**4**, p. 189].

complexity" despite "its extremely simple and complete mathematical theory" [**3**]. An extensive amount of material has been written about the name "Nim" and its possible origins. The most probable explanation is that it comes from the the German verb *nehmen*, imperative form *nimm*, meaning to take. Bouton had studied in Leipzig for three semesters with Sophus Lie.

A 16th century Italian origin

The oldest simplified variation of Nim found to date in Europe is in a manuscript of the Renaissance, *De Viribus Quantitatis* (On the powers of numbers) [**8**], a treatise written between 1496 and 1508 by the Italian mathematician Fra Luca Bartolomeo de Pacioli (1445–1517), one of the most famous mathematicians of his time. Pacioli falls in the tradition of the humanist tendency to mathematize the world and leave behind Aristotelian philosophy. The *De Viribus Quantitatis* can be considered to be one of the first works entirely devoted to mathematical recreations [**12**].

The manuscript, kept at the Bologna University Library, is written in Italian and consists of 309 sheets organized into three parts: 120 arithmetical recreations (*Delle forze naturali cioé de Arithmetica*), 139 problems dealing with geometry (*Della virtu et forza lineale et geometria*), and a few hundred proverbs, poems, riddles, and puzzles (*De documenti morali utilissimi*). One of Pacioli's key aims was to reveal the power of numbers and to demonstrate that they could be understood in a concrete way through card, dice, tarot, and board games.

Of interest here is Problem XXXIIII of the first part: "Finish any number before the opponent, without taking more than a certain finite number."

This rather unclear wording makes sense when Pacioli gives the solution: He suggests that two players reach the number 30 by, in turn, adding numbers ranging from 1 to 6. Pacioli justifies the number 6 as the highest number of points that can be reached with a single die. In fact, that is a simplified version of Bouton's Nim game, one that is played with only one pile consisting of 30 objects and in which each removal is limited to a maximum of 6 objects. We call this version additive because one adds numbers instead of reducing piles. Yet, the solution remains based on the same principle as Bouton's Nim: There exist safe combinations that secure the win, provided that one plays correctly. To distinguish this one pile version, we call these "safety steps."

Pacioli gives this winning strategy: Four safety steps have to be reached, namely 2, 9, 16, and 23. At this stage, Pacioli does not explain how he has determined these safety steps, but it seems obvious that he used backward induction as follows. On your ideal last move, you will add 1, 2, 3, 4, 5, or 6 and reach 30. What then should be your second to last move? You want a number from which your opponent cannot win

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but any move she makes allows you to win. This number is 23 since the result of your opponent's next move will be at least 24 and at most 29 and from any of those numbers you can win in your next move. Reasoning in the same way, the safety step before 23 is 16, then 9, and finally 2. So, you must manage to be the first to reach one of these safety steps and then continue up to 30.

Pacioli does not explain safety steps by this method, but he suggests a general method to find them for any game: "Always divide the number that you wish to arrive at by one more than has been taken and the remainder of the said division will always be the first [step of the] progression." If the division comes out exactly and the remainder is zero, Pacioli considers the case to be more difficult, and he clearly explains the backward induction in a particular example. He considers the case where the number to be reached is 35 by adding numbers between 1 and 6:

> For 35, take away 7, and 28 remains for the [step]; the other [step] takes away 7, and 21 remains; the other [step] 14; the other [step] 7. Therefore, he takes whatever he wants up to 6, and you will take, or actually you will make 7 the first degree, and then 14, 21, 28, and 35, and so on.

Pacioli distinguishes these two situation, but they can be analyzed the same way. Transcribed into modern notations, they both come down to the following result.

> Let n be the number to reach and k the maximum number that can be added in each turn. If dividing n by $k+1$ gives a remainder of zero, then $k+1$ is the first safety step, $2(k+1)$ the second, etc. If dividing n by $k+1$ gives a nonzero remainder, then the remainder is the first safety step and the others are determined by successively adding $k+1$ to this remainder.

It must be noted that Pacioli never gives this explanation by generalizing numbers and using variables. Indeed, it was not until Bouton's article that the solution to a combinatorial game was formalized. This is due to the relatively late development of algebra and the lack of an appropriate symbolism to represent unknown quantities and write equations. For centuries, clever arithmetical methods were used to solve problems that we would now tackle with algebra. Consequently, it was impossible then to obtain general conclusions. Rather than developing one formula that could handle each example with a variable, each case was studied individually.

Pacioli's manuscript was kept in the archives of the University of Bologna for nearly 500 years and was only published in 1997 (by Maria Garlaschi Peirani, who also transcribed it into modern Italian) but seems to have been consulted since the Middle Ages. Indeed, the 17th century was marked by the travels of scholars all around Europe, thus it is not surprising to find Pacioli's additive version of Nim in a French book.

To France and Germany in the 17th century

Pacioli's problem was to be found in *Problemes plaisans et delectables, qui se font par les nombres* (Pleasant and delectable problems, which are made by numbers) [**1**] by Claude-Gaspard Bachet (1581–1638), often considered to be the first (published) work on recreational mathematics. The fact that Pacioli was a major source of inspiration for Bachet has been studied in detail: About a third of the problems suggested by Bachet are directly linked to Pacioli. In the first edition of Bachet's 1612 book, Problem XIX offers a version of Nim that is similar to the one formulated by Pacioli.

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> If two persons have agreed to say one after another alternately a number which does not go beyond a certain precise number, and then add together the numbers they will say, who will reach the first another defined number; do it such that we always reach first the defined number.

Bachet studies the case when two players must reach 100 by adding numbers ranging from 1 to 10, "or any smaller number . . . such that the player who will say the number that achieves 100 is recognized the winner."

During the 17th century, there was a democratization of scholarship: More frequent visits to libraries by scientists, epistolary correspondence, and the increase of curiosity cabinets were all factors that helped facilitate the circulation and diffusion of knowledge. For example, the German professor of mathematics Daniel Schwenter (1585–1636) was given a 1624 collection by Jean Leurechon entitled *Récréation mathématique, composée de plusieurs problèmes plaisants et facétieux* (Mathematical recreation, made of several pleasant and mischievous problems) [**7**]. By the way, this was the first appearance of the words "mathematical recreation" (in the singular) in the title of a book. Schwenter, who did not know French, had the book translated and was greatly interested in Leurechon's problems. This led to Schwenter's *Deliciae Physico-Mathematicae* [**10**], which contains the additive ancestor of Nim: "Problem XLV: You both must count to 30. The winner is the one who first reaches 30. But it is not permitted to add more than 6 at each turn." The problem was not included in Leurechon's book; Schwenter explains that he found it in another 1624 collection, *Cryptomenytices et Cryptographiae Libri IX* by "Gustavus Selenus," a pseudonym of Augustus the Younger, Duke of Brunswick-Lüneburg.

18th century equestrians

During the 18th century, salons became important for discussions and exchanges of ideas between aristocrats and bourgeois. They were usually centered around games and amusements. Within this context, the French bookseller and publisher André-Joseph Panckoucke (1703–1753) proposed in 1749 a more fictionalized version of the additive Nim, where two horse riders decide to play a game while traveling [**9**]. The problem is called *Le Piquet des Cavaliers* (the horseriders' stake), and Piquet refers to a card game that at the time was considered one of the most dignified games, along with chess and backgammon. In the card game, two players take turns removing a card from the pack and adding its value to the sum already obtained from the previous draws. The two riders of our problem do not have any cards (which would not be very useful for riding) so they play orally.

In 1769, this equestrian problem was included in *Nouvelles récréations mathématiques et physiques* (New mathematical and physical recreations) [**5**] by Edmé-Gilles Guyot (1706–1786). He also republished Leurechon's *Récréation mathématique* in four volumes, the second of which was dedicated to arithmetical recreations.

Inspired by Leurechon and other French collections, the Englishman William Hooper wrote his 1774 collection *Rational Recreations* [**6**]. However, the additive version of Nim was named "The Magical Century" with no horses involved. Hooper's problem is interesting because the way he presents the solution is slightly different from those given above. Hooper begins with a reminder of multiplication by 11: "If the number 11 be multiplied by any one of the nine digits, the two figures of the product will always be similar." (See Figure 2.) Next, he chooses to add in turns counters piled up on a table until he obtains 100, without adding more than 10 counters at the same

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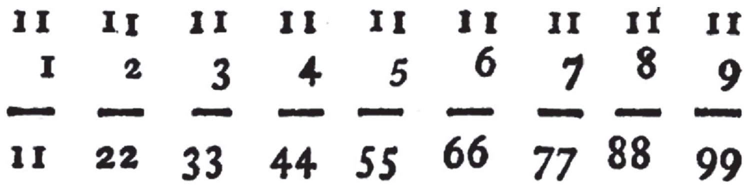

Figure 2. The multiplication table of the nine first nonzero natural numbers by 11 [**6**, p. 31].

time. It is worth noticing that Hooper opts for a more visual layout of the recreation, choosing to manipulate counters instead of adding abstract values in his head. This recreation is really closer to a game, including the handling of objects, than an exercise of mental arithmetic. We also point out a more direct link with the Nim of Bouton who considered piles of various-sized counters, which also had to be manipulated.

In 1820, John Badcock presented *A Curious Recreation with a Hundred Numbers*, usually called the *Magical Century* [**2**] in exactly the same way that Hooper did, at least at first. Badcock echoes the explanation using the multiplication table of the nine first nonzero natural numbers by 11; he also assumes that players handle counters and have to reach 100 without adding more than 10 at each turn. But, at the start, each player has 50 counters. This slight variation considerably changes the game and its solution! Indeed, if we start the game by playing one counter, and if our opponent adds only one counter in each of his turns, we will have to use 10 counters in each turn to reach the safety steps (12, 23, 34, 45, ...). Within five turns, we will already have taken 41 counters from our stock, compared with 5 used by our opponent, which makes a sum of 46. Unfortunately, Badcock only copies Hooper's solution, almost word for word, without noticing that his variation changes everything. He would have been better off copying the problem too.

From the beginning of the 19th century, this version of additive Nim could be found in numerous works of mathematical riddles and recreations in France, England, and Germany, sometimes presented as *Piquet sans cartes* (Piquet without cards) or simply as a two-player game. Some authors set out the game under a subtractive version (start at n and subtract at the most k in each turn), and a misère version appeared, in which the one who is the last to play is the loser [**13**].

We do not know which, if any, of these led to Nim. Bouton explains,

> The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it. [**3**]

Indeed, Fan-Tan is a very different game involving chance. (Bouton does credit Paul More for the misère version of Nim.) Bouton's proposed name has definitely taken hold and led to the rich theory of combinatorial games.

Acknowledgment. The translation of Pacioli's Effect XXXIIII was kindly given to me by David Singmaster during the Board Game Studies Colloquium XVI of April 2013. It is part of Singmaster's translation (draft) of *De Viribus Quantitatis*. I warmly thank him for this help.

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Summary. The first occurrence of Nim dates back to 1901 when the mathematician Charles Leonard Bouton published an article on its solution. But what are the origins of the Nim game? This article offers a survey of European ancestors of Nim.

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I'm not suggesting that we shouldn't entertain reforms to the redistricting process, such as having more independent commissions, and fewer incumbent legislators, draw the lines. But whoever draws the lines, there's no reason to draw straight ones. Representation is about people, not polygons.

—John Sides,
Monkey Cage blog,

http://www.washingtonpost.com/blogs/monkey-cage/wp/2014/05/08/why-weird-congressional-districts-can-be-good-congressional-districts/

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