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LEONARDO OF PISA AND HIS LIBER QUADRATORUM.

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The thirteenth century is a period of great fascination for the historian, whether his chief interest is in political, social, or intellectual movements. During this century great and far-reaching changes were taking place in all lines of human activity. It was the century in which culminated the long struggle between the Papacy and the Empire; it brought the beginnings of civil liberty in England; it saw the building of the great Gothic cathedrals, and the establishment and rapid growth of universities in Paris, Bologna, Naples, Oxford, and many other centers. The crusades had awakened the European peoples out of their lethargy of previous centuries, and had brought them face to face with the more advanced intellectual development of the East. Countless travelers passed back and forth between Italy and Egypt, Asia Minor, Syria, and Bagdad; and not a few adventurous and enterprising spirits dared to penetrate as far as India and China. The name of Marco Polo will occur to everyone, and he is only the most famous among many who in those stirring days truly discovered new worlds.

Among the many valuable gifts which the Orient transmitted to the Occident at this time, undoubtedly the most precious was its scientific knowledge, and in particular the Arabian and Hindu mathematics. The transfer of knowledge and ideas from East to West is one of the most interesting phenomena of this interesting period, and accordingly it is worth while to consider the work of one of the pioneers in this movement.

Leonardo of Pisa, known also as Fibonacci,¹ in the last years of the twelfth century made a tour of the East, saw the great markets of Egypt and Asia Minor,

¹ This is probably a contraction for "Filiolum Bonacci," or possibly for "Filius Bonacci"; that is, "of the family of Bonacci" or "Bonacci's son." See Boncompagni, *Della Vita e delle Opere di Leonardo Pisano, matematico del secolo decimoterzo*, Rome, 1852, pp. 8-12.

went as far as Syria, and returned through Constantinople and Greece.¹ Unlike most travelers, Leonardo was not content with giving a mere glance at the strange and new sights that met him, but he studied carefully the customs of the people, and especially sought instruction in the arithmetic system that was being found so advantageous by the Oriental merchants. He recognized its superiority over the clumsy Roman numeral system which was used in the West, and accordingly decided to study the Hindu-Arabic system thoroughly and to write a book which should explain to the Italians its use and applications. Thus the result of Leonardo's travels was the monumental *Liber Abaci* (1202), the greatest arithmetic of the middle ages, and the first one to show by examples from every field the great superiority of the Hindu-Arabic numeral system over the Roman system exemplified by Boethius.² It is true that Leonardo's *Liber Abaci* was not the first book written in Italy in which the Hindu-Arabic numerals were used and explained,³ but no work had been previously produced which in either the extent or the value of its contents could for a moment be compared with this. Even today it would be thoroughly worth while for any teacher of mathematics to become familiar with many portions of this great work. It is valuable reading both on account of the mathematical insight and originality of the author, which constantly awaken our admiration, and also on account of the concrete problems, which often give much interesting and significant information about commercial customs and economic conditions in the early thirteenth century.

Besides the *Liber Abaci*, Leonardo of Pisa wrote an extensive work on geometry, which he called *Practica Geometriæ*. This contains a wide variety of interesting theorems, and while it shows no such originality as to enable us to rank Leonardo among the great geometers of history, it is excellently written, and the rigor and elegance of the proofs are deserving of high praise. A good idea of a small portion of the *Practica Geometriæ* can be obtained from Archibald's very successful restoration of Euclid's *Divisions of Figures*.⁴

The other works of Leonardo of Pisa that are known are *Flos*, a *Letter to Magister Theodorus*, and the *Liber Quadratorum*. These three works are so original and instructive, and show so well the remarkable genius of this brilliant mathematician of the thirteenth century, that it is highly desirable that they be made available in English translation. It is my intention to publish such a translation when conditions are more favorable, but in the meantime a short account of the *Liber Quadratorum* will bring to those whose attention has not yet been called to it some idea of the interesting and valuable character of the book.

The *Liber Quadratorum* is dedicated to the Emperor Frederick II, who

¹ *Scritti di Leonardo Pisano*, 2 vols., Rome, 1857-61. Vol. I, p. 1.

² Boethius, ed. Friedlein, Leipzig, 1867. The arithmetic occupies pages 1-173. This was the arithmetic that was very generally taught throughout Europe before the thirteenth century, and its use continued to be widespread long after better works were in the field.

³ Smith and Karpinski, *The Hindu-Arabic Numerals*, Boston and London, 1911. Chapter VII gives an account of the first European writings on these numerals.

⁴ Archibald, *Euclid's Book on Divisions of Figures; with a Restoration based on Woepcke's Text and on the Practica Geometriæ of Leonardo Pisano*, Cambridge, England, 1915.

throughout his whole career showed a lively and intelligent interest in art and science, and who had taken favorable notice of Leonardo's *Liber Abaci*. In the dedication, dated in 1225, Leonardo relates that he had been presented to the Emperor at court in Pisa, and that Magister Johannes of Palermo had there proposed a problem¹ as a test of Leonardo's mathematical power. The problem was, to find a square number which when either increased or diminished by 5 should still give a square number as result. Leonardo gave a correct answer, $11\frac{97}{144}$. For $11\frac{97}{144} = (3\frac{5}{2})^2$, $6\frac{97}{144} = (2\frac{7}{2})^2$, and $16\frac{97}{144} = (4\frac{1}{2})^2$. Through considering this problem and others allied to it, Leonardo was led to write the *Liber Quadratorum*.² It should be said that this problem had been considered by Arab writers with whose works Leonardo was unquestionably familiar; but his methods are original, and our admiration for them is not diminished by careful study of what had been done by his Arabian predecessors.³

In the *Liber Quadratorum*, Leonardo has given us a well-arranged, brilliantly-written collection of theorems from indeterminate analysis involving equations of the second degree. Many of the theorems themselves are original, and in the case of many others the proofs are so. The usual method of proof employed is to reason upon general numbers, which Leonardo represents by line segments. He has, it is scarcely necessary to say, no algebraic symbolism, so that each result of a new operation (unless it be a simple addition or subtraction) has to be represented by a new line. But for one who had studied the "geometric algebra" of the Greeks, as Leonardo had, in the form in which the Arabs used it,⁴ this method offered some of the advantages of our symbolism; and at any rate it is marvelous with what ease Leonardo keeps in his mind the relation between two lines and with what skill he chooses the right road to bring him to the goal he is seeking.

To give some idea of the contents of this remarkable work, there follows a list of the most important results it contains. The numbering of the propositions is not found in the original.

PROPOSITION I. THEOREM. Every square number⁵ can be formed as a sum of successive odd numbers beginning with unity. That is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

PROPOSITION II. PROBLEM. To find two square numbers whose sum is a square number. "I take any odd square I please, . . . and find the other from

¹ In the introduction to *Flos* we are told that two other problems were propounded at the same time. *Scritti*, II, p. 227.

² *Scritti*, II, p. 253.

³ See, for example, Woepcke, *Recherches sur plusieurs ouvrages de Leonard de Pise, et sur les rapports qui existent entre ces ouvrages et les travaux mathématiques des Arabes*, Rome, 1859.

⁴ Heath, T. L., *The Thirteen Books of Euclid's Elements*, Cambridge, 1908. Vol. I, pp. 372-374, 383-385, 386-388; Zeuthen, H. G., *Geschichte der Mathematik im Altertum und Mittelalter*, Copenhagen, 1896, pp. 44-53; Karpinski, L. C., *Robert of Chester's Latin translation of the Algebra of Al-Khwarizmi*, New York, 1915, pp. 77-89.

⁵ Throughout this article, unless otherwise stated, the word "number" is to be understood as meaning "positive integer."

the sum of all the odd numbers from unity up to that odd square itself."¹ Thus, if $2n + 1$ is a square ($= x^2$) then

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) + x^2 &= n^2 + (2n + 1) \\ &= \text{a sum of two squares} = (n + 1)^2 \end{aligned}$$

This is equivalent to Pythagoras's rule for obtaining rational right triangles, as stated by Proclus,² viz.,

$$\left(\frac{x^2 - 1}{2}\right)^2 + x^2 = \left(\frac{x^2 + 1}{2}\right)^2.$$

For, inasmuch as $2n + 1 = x^2$, we have $n = \frac{x^2 - 1}{2}$ and $n + 1 = \frac{x^2 + 1}{2}$.

PROPOSITION III. THEOREM. $\left(\frac{n^2}{4} - 1\right)^2 + n^2 = \left(\frac{n^2}{4} + 1\right)^2$.

This enables us to obtain rational right triangles in which the hypotenuse exceeds one of the legs by 2. It is attributed by Proclus to Plato². Leonardo also gives the rule in case the hypotenuse is to exceed one leg by 3, and indicates what the result would be if the hypotenuse exceeds one leg by any number whatever.

PROPOSITION IV. THEOREM. "Any square exceeds the square which immediately precedes it by the amount of the sum of their roots." That is, $n^2 - (n - 1)^2 = n + (n - 1)$. It follows from this that when the sum of two consecutive numbers is a square number, then the square of the greater will equal the sum of two squares. For, if $n + (n - 1) = u^2$, then $n^2 - (n - 1)^2 = u^2$ or $n^2 = u^2 + (n - 1)^2$.

PROPOSITION V. PROBLEM. Given $a^2 + b^2 = c^2$, to find two integral or fractional numbers x, y , such that $x^2 + y^2 = c^2$. Solution: Find two other numbers m and n such that³ $m^2 + n^2 = q^2$. If $q^2 \neq c^2$, multiply the preceding equation by c^2/q^2 , obtaining

$$\left(\frac{c}{q} \cdot m\right)^2 + \left(\frac{c}{q} \cdot n\right)^2 = c^2$$

so that $x = c/q \cdot m$, $y = c/q \cdot n$ is a solution.

PROPOSITION VI. THEOREM. "If four numbers not in proportion are given, the first being less than the second, and the third less than the fourth, and if the sum of the squares of the first and second is multiplied by the sum of the squares of the third and fourth, there will result a number which will be equal in two ways to the sum of two square numbers." That is,

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ad + bc)^2 + (ac - bd)^2.$$

This very important theorem should be called Leonardo's Theorem, for it is⁸

¹ The use of quotation marks indicates a literal translation of Leonardo's words; in other cases the exposition follows his thought without adhering closely to his form of expression.

² Proclus, ed. Friedlein, Leipzig, 1873, p. 428.

³ This is possible by Proposition II or Proposition III.

not found definitely stated, to say nothing of being proved, in any earlier work. Leonardo considers also the case where a, b, c , and d are in proportion, and shows that then $(a^2 + b^2) \cdot (c^2 + d^2)$ is equal to a square and the sum of two squares. This gives him still another way of finding rational right triangles.¹

PROPOSITION VII. THEOREM. $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$.² Leonardo proves this very simply as a corollary of Proposition VI.

PROPOSITION VIII. PROBLEM. "To find two numbers the sum of whose squares is a number, not a square, formed from the addition of two given squares." That is, to find x and y such that $x^2 + y^2 = a^2 + b^2$. Choose any two numbers c and d , such that $c^2 + d^2$ is a square, and write $(a^2 + b^2)(c^2 + d^2)$ as a sum of two squares, let us say $p^2 + q^2$; this we can do by Proposition VI. Construct the right triangle whose legs are p and q ; then the similar triangle whose hypotenuse is equal to $\sqrt{c^2 + d^2}$ will have as its legs the two required numbers x and y .

PROPOSITION IX. THEOREM.

$$6(1^2 + 2^2 + 3^2 + \dots + n^2) = n(n + 1)(2n + 1).$$

The proof of this is strikingly original, and proceeds from the identity

$$n(n + 1)(2n + 1) = n(n - 1)(2n - 1) + 6n^2.$$

Hence

$$n(n - 1)(2n - 1) = (n - 1)(n - 2)(2n - 3) + 6(n - 1)^2,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$2 \cdot 3 \cdot (2 + 3) = 1 \cdot 2 \cdot (1 + 2) + 6 \cdot 2^2,$$

$$1 \cdot 2 \cdot (1 + 2) = \qquad \qquad \qquad 6 \cdot 1^2.$$

It follows by addition that

$$n(n + 1)(2n + 1) = 6(1^2 + 2^2 + 3^2 + \dots + (n - 1)^2 + n^2).$$

PROPOSITION X. THEOREM.

$$12[1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2] = (2n - 1)(2n + 1)4n.$$

Leonardo gives a proof very similar to that of Proposition IX.

PROPOSITION XI. THEOREM.

$$12[2^2 + 4^2 + 6^2 + \dots + (2n)^2] = 2n(2n + 2)(4n + 2),$$

and likewise

$$18[3^2 + 6^2 + 9^2 + \dots + (3n)^2] = 3n(3n + 3)(6n + 3),$$

and

$$24[4^2 + 8^2 + 12^2 + \dots + (4n)^2] = 4n(4n + 4)(8n + 4),$$

¹ For instance, letting $a = 6, b = 4, c = 3, d = 2$,

$(36 + 16)(9 + 4) = 676 = (6 \cdot 3 + 4 \cdot 2)^2 = (6 \cdot 2 + 4 \cdot 3)^2 + (6 \cdot 3 - 4 \cdot 2)^2 = 26^2 = 24^2 + 10^2.$

² This is Euclid's general solution of the problem of finding rational right triangles. Heath, *op. cit.*, III, p. 63. (Euclid's Elements, X, Lemma to Theorem 29.)

and in general

$$6a[a^2 + (2a)^2 + (3a)^2 + \cdots + (na)^2] = na(na + a)(2na + a).$$

Here Leonardo has almost discovered the general result

$$\begin{aligned} a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + [a + (n - 1)d]^2 \\ = \frac{6na^2 + 6n(n - 1)ad + n(n - 1)(2n - 1)d^2}{6}. \end{aligned}$$

His method needed no change at all, in fact.

PROPOSITION XII. THEOREM. If $x + y$ is even, $xy(x + y)(x - y)$ is divisible by 24; and in any case $4xy(x + y)(x - y)$ is divisible by 24. A number of this form is called by Leonardo a *congruum*, and he proceeds to show that it furnishes the solution to a problem proposed by Johannes of Palermo.

PROPOSITION XIII. PROBLEM. "To find a number which, being added to, or subtracted from, a square number, leaves in either case a square number." Leonardo's solution of this, the problem which had stimulated him to write the *Liber Quadratorum*, is so very ingenious and original that it is a matter of regret that its length prevents its inclusion here. It is not too much to say that this is the finest piece of reasoning in number theory of which we have any record, before the time of Fermat. Leonardo obtains his solution by establishing the identities

$$(x^2 + y^2)^2 - 4xy(x^2 - y^2) = (y^2 + 2xy - x^2)^2$$

and

$$(x^2 + y^2)^2 + 4xy(x^2 - y^2) = (x^2 + 2xy - y^2)^2.$$

PROPOSITION XIV. PROBLEM. To find a number of the form $4xy(x + y)(x - y)$ which is divisible by 5, the quotient being a square. Take $x = 5$, and y equal to a square such that $x + y$ and $x - y$ are also squares. The least possible value for y is 4, in which case

$$4xy(x + y)(x - y) = 4 \cdot 5 \cdot 4 \cdot 9 \cdot 1 = 720.$$

PROPOSITION XV. PROBLEM. "To find a square number which, being increased or diminished by 5, gives a square number. Let a congruum be taken whose fifth part is a square, such as 720, whose fifth part is 144; divide by this the squares congruent to 720,¹ the first of which is 961, the second 1681, and the third 2401. The root of the first square is 31, of the second is 41, and of the third is 49. Thus there results for the first square $6\frac{9}{144}$, whose root is $2\frac{7}{12}$, which results from the division of 31 by the root of 144, that is, by 12; and for the second, that is, for the required square, there will result $11\frac{9}{144}$, whose root is $3\frac{5}{12}$, which results from the division of 41 by 12; and for the last square there will result $16\frac{9}{144}$, whose root is $4\frac{1}{2}$."

¹ That is, the three squares in arithmetic progression, whose common difference is the congruum 720. They are obtained by Proposition XIII, thus: Taking $x = 5$ and $y = 4$, $y^2 + 2xy - x^2 = 31$, the root of the first square; $x^2 + y^2 = 41$, the root of the second square; and $x^2 + 2xy - y^2 = 49$, the root of the third square.

PROPOSITION XVI. THEOREM. When $x > y$, $(x + y)/(x - y) \neq x/y$. It follows that $x(x - y)$ is not equal to $y(x + y)$, and "from this," Leonardo says, "it may be shown that no square number can be a congruum." For if $xy(x + y)(x - y)$ could be a square, either $x(x - y)$ must be equal to $y(x + y)$, which this proposition proves to be impossible, or else the four factors must severally be squares, which is also impossible. Leonardo to be sure overlooked the necessity of proving this last assertion, which remained unproved until the time of Fermat.¹

PROPOSITION XVII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + x = u^2,$$

$$x^2 - x = v^2.$$

The solution is obtained by means of any set of three squares in arithmetic progression, that is, by means of Proposition XIII. Let us take x_1^2 , x_2^2 , and x_3^2 for the three squares, and let the common difference, that is, the congruum, be d . Leonardo says that the solution of the problem is obtained by giving x the value x_2^2/d . For then

$$x^2 + x = \frac{x_2^4}{d^2} + \frac{x_2^2}{d} = \frac{x_2^2(x_2^2 + d)}{d^2} = \frac{x_2^2 x_3^2}{d^2};$$

and

$$x^2 - x = \frac{x_2^4}{d^2} - \frac{x_2^2}{d} = \frac{x_2^2(x_2^2 - d)}{d^2} = \frac{x_2^2 x_1^2}{d^2}.$$

PROPOSITION XVIII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + 2x = u^2,$$

$$x^2 - 2x = v^2.$$

The method is similar to that in Proposition XVII, the value of x being found to be $2x_2^2/d$. Leonardo adds, "You will understand how the result can be obtained in the same way if three or more times the root is to be added or subtracted."

PROPOSITION XIX. PROBLEM. To solve (in integers) the pair of equations

$$x^2 + y^2 = u^2,$$

$$x^2 + y^2 + z^2 = v^2.$$

Take for x and y any two numbers that are prime to each other and such that the

¹ Fermat, *Oeuvres*, Paris, 1891, vol. 1, p. 340; Heath, *Diophantus of Alexandria*, Cambridge, 1910, p. 293.

² The simplest numerical example would be $x_1^2 = 1$, $x_2^2 = 25$, $x_3^2 = 49$, and this is the illustration given by Leonardo. It leads to $x = \frac{25}{4}$, from which we have $x^2 + x = \frac{1^2 25^2}{4^2} = (\frac{35}{4})^2$ and $x^2 - x = \frac{25^2}{4^2} = (\frac{5}{2})^2$.

sum of their squares is a square, let us say u^2 . Adding all the odd numbers from unity to $u^2 - 2^1$, the result is $((u^2 - 1)/2)^2$.

Now

$$\left(\frac{u^2 - 1}{2}\right)^2 + u^2 = \left(\frac{u^2 + 1}{2}\right)^2.$$

Thus

$$z^2 = \left(\frac{u^2 - 1}{2}\right)^2,$$

and

$$v^2 = \left(\frac{u^2 + 1}{2}\right)^2.$$

PROPOSITION XX. PROBLEM. To solve in rational numbers the set of equations

$$x + y + z + x^2 = u^2,$$

$$x + y + z + x^2 + y^2 = v^2,$$

$$x + y + z + x^2 + y^2 + z^2 = w^2.$$

By an extension of the method used in Proposition XIX Leonardo obtains the results $x = 3\frac{1}{5}$, $y = 9\frac{2}{5}$, $z = 28\frac{4}{5}$. He even goes farther and obtains the integral solutions $x = 35$, $y = 144$, $z = 360$. He continues, "And not only can three numbers be found in many ways by this method but also four can be found by means of four square numbers, two of which in order, or three, or all four added together make a square number. . . . I found these four numbers, the first of which is 1295, the second 4566 $\frac{6}{7}$, the third 11417 $\frac{1}{7}$, and the fourth 79920." In the midst of the explanation of how these values were obtained, the MS. of the *Liber Quadratorum* breaks off abruptly. It is probable, however, that the original work included little more than what the one known Ms. gives. At all events, considering both the originality and power of his methods, and the importance of his results, we are abundantly justified in ranking Leonardo of Pisa as the greatest genius in the field of number theory who appeared between the time of Diophantus and that of Fermat.

¹ Here u^2 is odd, because it is the sum of the squares of two numbers x and y which are prime to each other. It is not possible that both x and y are odd, since $(2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2$, and this is divisible by 2 but not by 4, and hence can not be a square. Thus, of the numbers x and y , one must be even and the other odd, hence $x^2 + y^2$ is odd.