ARTICLES

Liu Hui and the First Golden Age of Chinese Mathematics

PHILIP D. STRAFFIN, JR.
Beloit College
Beloit, WI 53511

Introduction

Very little is known of the life of Liu Hui, except that he lived in the Kingdom of Wei in the third century A.D., when China was divided into three kingdoms at continual war with one another. What is known is that Liu was a mathematician of great power and creativity. Liu's ideas are preserved in two works which survived and became classics in Chinese mathematics. The most important of these is his commentary, dated 263 A.D., on the *Jiuzhang suanshu*, the great problem book known in the West as the *Nine Chapters on the Mathematical Art*. The second is an independent work on mathematics for surveying, the *Haidao suanjing*, known as the *Sea Island Mathematical Manual.*

In this paper I would like to tell you about some of the remarkable results and methods in these two works. I think they should be more widely known, for several reasons. First, we and our students should know more about mathematics in other cultures, and we are probably less familiar with Chinese mathematics than with the Greek, Indian, and Islamic traditions more directly linked to the historical development of modern mathematics. Second, Western mathematicians who do know something about the Chinese tradition often characterize Chinese mathematics as calculational and utilitarian rather than theoretical. Chinese mathematicians, it is said, developed clever methods, but did not care about mathematical justification of those methods. For example,

> Mathematics was overwhelmingly concerned with practical matters that were important to a bureaucratic government: land measurement and surveying, taxation, the making of canals and dikes, granary dimensions, and so on... Little mathematics was undertaken for its own sake in China. [**2**, p. 26]

While there is justice in this generalization, Liu Hui and his successors Zu Chongzhi and Zu Gengzhi were clearcut exceptions. Their methods were different from those of the Greeks, but they gave arguments of cogency and clarity which we can honor today, and some of those arguments involved infinite processes which we recognize as underlying the integral calculus.

My final reason is that I think mathematical genius should be honored wherever it is found. I hope you will agree that Liu Hui is deserving of our honor.

To understand the context of Liu's work, we must first consider the state of Chinese mathematical computation in the third century A.D. We will then look at the general nature of the *Nine Chapters* and Liu's commentary on it, and at Liu's *Sea Island Mathematical Manual.* I will then focus on three of Liu's most remarkable achievements in geometry—his calculation of π, his derivation of the volume of pyramidal solids, and his work on the volume of a sphere and its completion by Zu Gengzhi.

Chinese Calculation in the First Century A.D.

From at least the period of the Warring States (475–221 B.C.) a base ten positional number system was in common use in China **[12]**. Calculations were done using rods made from bone or bamboo, on a counting board marked off into squares. The numerals from 1 to 9 were represented by rods, as in FIGURE 1. Their placement in squares, from left to right, represented decreasing powers of ten. Rods representing odd powers of ten were rotated 90° for clarity in distinguishing the powers. A zero was represented simply by a blank square, called a *kong*, where the marking into squares prevented the ambiguity sometimes present in, say, the Babylonian number system.

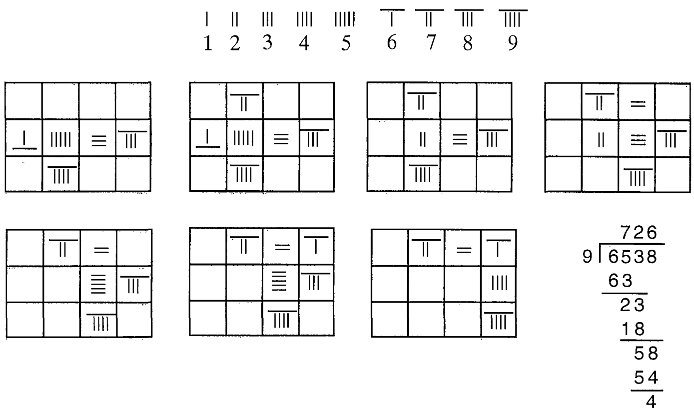


FIGURE 1
Numerals and the division algorithm.

There were efficient algorithms for addition, subtraction, multiplication, and division. For example, the division algorithm is shown in FIGURE 1, except that you should imagine the operations being done rapidly with actual sticks. Notice the close relationship to our modern long division algorithm, although subtraction is easier because sticks are physically removed. In fact, it is identical to the division algorithm given by al-Khwarizmi in the ninth century and later transmitted to Europe, raising the complicated problem of possible transmission through India to the West **[12]**. (See **[17]** for a conservative discussion.)

Notice how the answer $726\frac{4}{9}$ ends up with 726 in the top row, and then 4 above 9. This led Chinese calculators to represent fractions by placing the numerator above the denominator on the counting board. By the time of the *Nine Chapters* there was a completely developed arithmetic of fractions: they could be multiplied, divided, compared by cross multiplication, and reduced to lowest form using the "Euclidean algorithm" to find the largest common factor of the numerator and denominator. Addition was performed as $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, and then the fraction was reduced if necessary. In the *Nine Chapters*, 160 of the 246 problems involve computations with fractions **[11]**.

We will see that Chapter Eight of the *Nine Chapters* solves systems of linear equations by the method known in the West as "Gaussian Elimination" after C. F. Gauss (1777–1855), which, of course, involves subtracting one row of numbers from another. In the course of such calculations, it is inevitable that negative numbers will arise. This presented no problems to Chinese calculators: two colors of rods were used, and correct rules were given for manipulating the colors. Liu Hui suggested in his commentary on the *Nine Chapters* that negative numbers be treated abstractly:

> When a number is said to be negative, it does not necessarily mean that there is a deficit. Similarly, a positive number does not necessarily mean that there is a gain. Therefore, even though there are red (positive) and black (negative) numerals in each column, a change in their colors resulting from the operations will not jeopardize the calculation. [**17**, pp. 201–202]

Perhaps most remarkably, Chinese mathematicians had developed by the time of the *Nine Chapters* efficient algorithms for computing square roots and cube roots of arbitrarily large numbers. The algorithm for the square root computed the root digit by digit, by the same method which used to be taught in American schools before the coming of the calculator. Martzloff [**17**] works through an example, and Lam [**11**] shows how it would look on a counting board. The algorithm for finding cube roots was similar, although, of course, more complicated.

In other words, by the time of the *Nine Chapters* the Chinese had developed a number system and a collection of calculational algorithms essentially equivalent to our modern system, with the exception of decimal fractions.

Nine Chapters on the Mathematical Art

Nine Chapters on the Mathematical Art is a compilation of 246 mathematical problems loosely grouped in nine chapters. Some of its material predates the great book-burning and burial-alive of scholars of 213 B.C., ordered by emperor Shih Huang-ti of the Qin dynasty. Indeed, Liu Hui writes in the preface of his commentary:

> In the past, the tyrant Qin burnt written documents, which led to the destruction of classical knowledge... Because of the state of deterioration of the ancient texts, Zhang Cang and his team produced a new version... filling in what was missing. [**17**, p. 129]

It is believed that the *Nine Chapters* were put in their final form sometime before 100 A.D. It "became, in the Chinese tradition, the mandatory reference, the classic of classics." [**17**, p. 14] At the time of this writing there is no complete English translation of the *Nine Chapters*, although there are many scholarly Chinese editions, and translations into Japanese, German, and Russian. An English translation by J. N. Crossley and Shen Kangsheng is in preparation, to be published by Springer-Verlag. For summaries, see [**11**], [**17**], [**18**], [**21**].

The format of the *Nine Chapters* is terse: a problem, its answer, and a recipe for obtaining the answer. Usually no justification is given for the method of solution. Just the facts.

Chapter One has many problems on the arithmetic of fractions, and a section on computing areas of planar figures, with correct formulas for rectangles, triangles, and trapezoids. Here's a problem on the area of a circle:

1.32: There is a circular field, circumference 181 *bu* and diameter $60\frac{1}{3}$ *bu*. Find the area of the field.

Answer: 11 *mu* $90\frac{1}{12}$ *bu*. (1 *mu* = 240 *bu*)

Method: Mutually multiply half of the circumference and half of the diameter to obtain the area in *bu*. Or multiply the diameter by itself, then by 3 and divide by 4. Or multiply the circumference by itself and divide by 12. [**11**, p. 13]

The first method is correct, but the data of the problem and the other two methods assume that the ratio of the circumference of a circle to its diameter, which we call π, is three. This assumption is made throughout the *Nine Chapters*.

Chapter Two is a series of commodity exchange problems involving proportions. Chapter Three concerns problems of "fair division." The solutions given may not seem very fair to us:

3.8: There are five persons: Dai Fu, Bu Geng, Zan Niao, Shang Zao, and Gong Shi. They pay a total of 100 *qian*. A command desired that the highest rank pays the least, and the successive ones gradually more. Find the amount each has to pay.

Answer: Dai Fu pays $8\frac{104}{137}$ *qian*; Bu Geng pays $10\frac{130}{137}$ *qian*; Zan Niao pays $14\frac{82}{137}$ *qian*; Shang Zao pays $21\frac{123}{137}$ *qian*; Gong Shi pays $43\frac{109}{137}$ *qian*. [**11**, p. 21]

The method calls for dividing the cost in proportions $\frac{1}{5}:\frac{1}{4}:\frac{1}{3}:\frac{1}{2}:1$, which gives practice in adding fractions, but badly exploits the lowest rank person!

Chapter Four contains problems asking for the calculation of square roots and cube roots. The last problem of Chapter Four is

4.24: There is a sphere of volume 16441866437500 *chi*. Find the diameter.

Answer: 14300 *chi*.

Method: Put down the volume in *chi*, multiply by 16 and divide by 9. Extract the cube root of the result to get the diameter of the sphere. [**11**, p. 23]

This gives the formula $V=\frac{9}{16}d^3$ for the volume of a sphere in terms of its diameter, which isn't correct even if we take $\pi=3$.

Chapter Five asks for the volumes of a number of solids, including several different kinds of pyramids, frustums of pyramids, cones and their frustums, and a wedge with a trapezoidal base. The given formulas are all correct, but no hint is given of how they were derived.

Chapter Six deals with fair division in a much more realistic way than the problems in Chapter Three. There are problems on transporting grain, taxation, and irrigation. There are also some less realistic problems which make one wonder how Chinese students must have felt about "word problems":

6.14: There is a rabbit which walks 100 *bu* before it is chased by a dog. When the dog has gone 250 *bu*, it stops and is 30 *bu* behind the rabbit. If the dog did not stop, find how many more *bu* it would have to go before it reaches the rabbit.

Answer: $107\frac{1}{7}$ *bu*. [**11**, p. 28]

Chapter Seven has a number of problems involving two linear equations in two unknowns, usually solved by the method of "false position." Problems in Chapter Eight involve solving n linear equations in n unknowns for n up to 5. The method of solution, described in detail, is Gaussian elimination on the appropriate matrix represented on the counting board. The Chinese called this method *fangcheng*. See [**17**] for an extended example. Perhaps the most interesting problem is

8.13: There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.

Answer: The well is 721 *cun* deep. A's rope is 265 *cun* long. B's rope is 191 *cun* long. C's rope is 148 *cun* long. D's rope is 129 *cun* long. E's rope is 76 *cun* long. [**11**, p. 37]

Notice that this problem involves five equations and six unknowns, and thus is indeterminate. Liu Hui pointed out that the solution gives only the necessary proportions for the lengths. It is also the smallest solution in integer lengths.

The problems in Chapter Nine involve right triangles and the "Pythagorean" theorem, which had long been independently known in China, where it was called the *gou-gu* theorem **[26]**. No proof is given of this theorem, or of a correct formula for the diameter of the inscribed circle in a right triangle. Similar right triangles are used to solve surveying problems involving one unknown distance or length.

Liu Hui's Commentary

The *Nine Chapters* presents its solution methods without justification. Liu Hui in his commentary set himself the goal of justifying those methods. One reason was practical, as Liu wrote about the *Nine Chapters*' use of 3 for the ratio of the circumference of a circle to its diameter:

> Those who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification it is very difficult to separate truth from fallacy. **[20**, p. 349]

Another reason has to do with seeing and appreciating the logical structure of mathematics:

> Things are related to each other through logical reasons so that like branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidate by prose and illustrate by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigor. **[20**, p. 355]

In this section, we'll begin our examination of Liu's attempt to attain "clarity as well as rigor" by looking at five of his contributions.

Problems in Chapter Four of the *Nine Chapters* require taking square roots using the square root algorithm. To take the square root of a $2k+1$ or $2k+2$ digit number N, the algorithm begins by finding the largest number $A_0 = a_0 \times 10^k$, where a_0 is a digit, such that $A_0^2 \leq N$. Then compute $N_1 = N - A_0^2$. Now find the largest $A_1 = a_1 \times 10^{k-1}$ such that $A_1(2A_0 + A_1) \leq N_1$, and form $N_2 = N_1 - A_1(2A_0 + A_1)$. Continue in this manner. If N is a perfect square, its square root will be the $(k+1)$-digit number $S = a_0 a_1 \cdots a_k$.

Liu Hui first gives a geometric argument, similar to arguments used in Greek geometric algebra, to explain why the algorithm works. Consider FIGURE 2, which is not to scale. (Liu's original figures were all lost, but most of them are easy to reconstruct from his verbal descriptions.) From a square of area N, we first subtract a square of side A_0, then the L-shaped figure of width A_1, which the Greeks called a gnomon, then a gnomon of width A_2, and so on until we exhaust the square.

Well, at least we exhaust the square if N is a perfect square, as it is in many of the *Nine Chapters* problems. (Some of the problems involve rational perfect squares, for instance $N = 564752\frac{1}{4}$ in problem 4.15.) But Liu also asks what happens if N is not a perfect square: "In this case it is not sufficient to say what the square root is about by simply ignoring the [remaining] gnomon." **[7**, p. 211] For integral but non-square N,

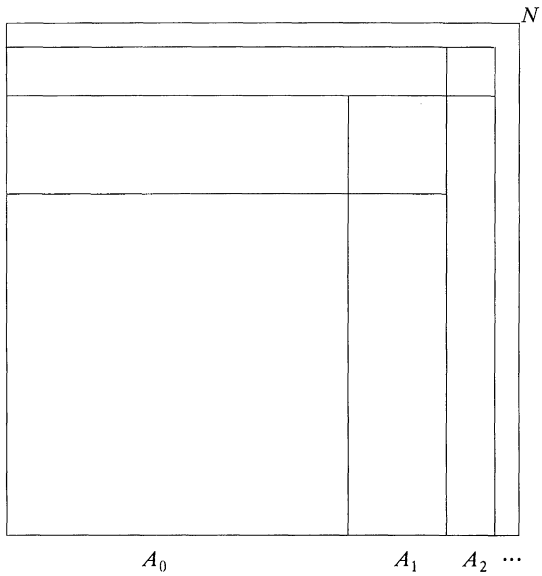


FIGURE 2
Geometry of the square root algorithm.

the square root algorithm yields $N = S^2 + R$, where $0 < R < 2S + 1$. Liu gives two ways of approximating the square root. The first is to take a rational approximation using

$$S + \frac{R}{2S+1} < \sqrt{N} < S + \frac{R}{2S}. \quad \textbf{[17]}$$

The second is even more interesting. If we continue the algorithm on the counting board past the last digit of N, we get

$$\sqrt{n} \approx a_0 a_1 \dots a_k + \frac{a_{k+1}}{10} + \frac{a_{k+2}}{100} + \dots.$$

The ancient Chinese had names for the fractions $1/10^k$ for k up to five. Liu suggests continuing the calculation down to "those small numbers for which the units do not have a name," and if necessary adding a fraction to a_{k+5} to get even greater accuracy **[11]**. In other words, it is not stretching very much to say that Liu Hui invented decimals; he certainly invented their calculational equivalent. We will see that he needed this kind of accuracy for his calculation of π. Liu also gave a justification for the cube root algorithm using a three-dimensional figure similar to Figure 2.

Chapter Eight of the *Nine Chapters* solved systems of linear equations using the *fangcheng* method on a counting board matrix: multiples of rows (actually columns, since the equations were set up vertically on the counting board) were systematically subtracted from other rows to reduce the matrix to triangular form. Liu Hui explains that the goal of this method is to reduce to a minimum the number of computations needed to find the solution: "generally, the more economic a method is, the better it is." In fact, Liu compares two different *fangcheng* methods for solving problem 8.18 by counting the number of counting board operations needed in each method **[17]**. Surely this is the first example in history of an operation count to compare the computational efficiency of two algorithms.

Finally, Chapter Nine of the *Nine Chapters* presented, without justification, solutions to a number of problems involving right triangles. Liu Hui justified these solutions by a series of ingenious "dissection" arguments, based on the principles that congruent figures have the same area, and that if we dissect a figure into a finite number of pieces, its area is the sum of the areas of the pieces. I'll give two examples.

The solution to problem 9.16 finds the diameter d of a circle inscribed in a right triangle with legs a and b and hypotenuse c by

$$d = \frac{2ab}{a+b+c}.$$

Liu's dissection proof of this result can be reconstructed as in FIGURE 3 [20]. See it?

For the second example, consider the famous *gou-gu* theorem that for a right triangle as above, $a^2 + b^2 = c^2$. For this theorem, Liu's verbal description of his proof is as follows:

> The shorter leg multiplied by itself is the red square, and the longer leg multiplied by itself is the blue square. Let them be moved about so as to patch each other, each according to its type. Because the differences are completed, there is no instability. They form together the area of the square on the hypotenuse. [31, p. 71]

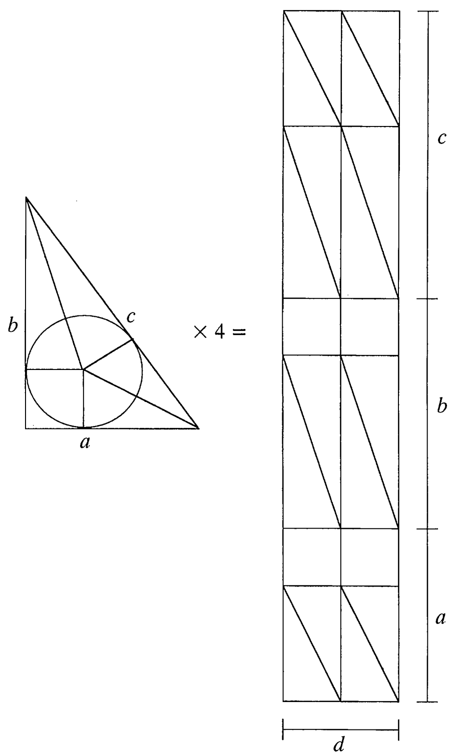


FIGURE 3
Diameter of a circle inscribed in a right triangle.

Clearly, Liu had a dissection proof of the *gou-gu* theorem. Just as clearly, the verbal description does not enable us to reconstruct Liu's diagram. FIGURE 4 shows two

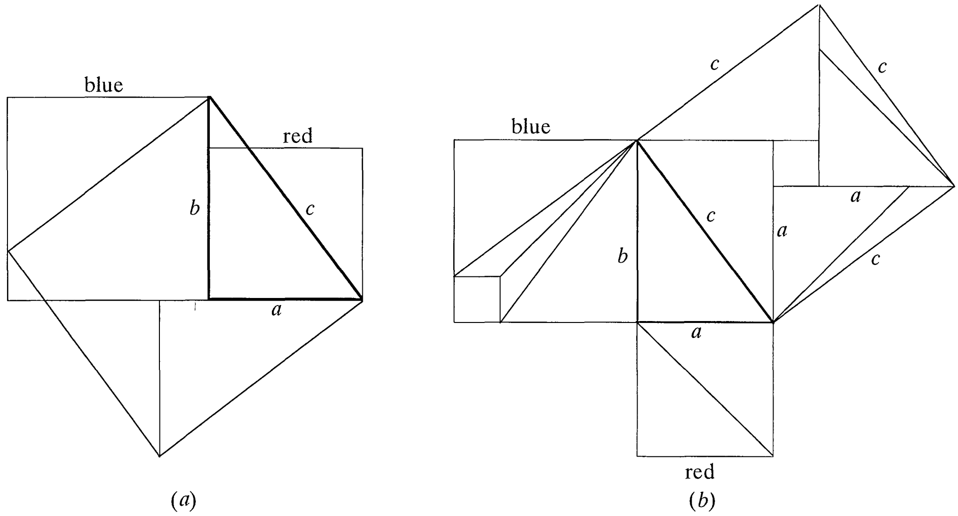


FIGURE 4
Dissection proofs of the *gou-gu* theorem.

proposed constructions. The first, where the square on the hypotenuse is allowed to overlap the squares on the legs, is due to Gu Guanguang in 1892, reported in [**17**]. The second, less straightforward but without overlapping squares, is from [**31**].

The Sea Island Mathematical Manual

Chapter Nine of the *Nine Chapters* included surveying problems involving one unknown distance or length. However, most real surveying problems involve several such unknowns. For example, we might wish to determine the height of, and distance to, a mountain which is inaccessible, perhaps because it is on an island we cannot reach. Liu Hui pointed out that we can do this by making two observations, and worked out the geometry of how to make two observations yield the unknown distances. If we wish also to know the height of a pine tree on top of that inaccessible mountain, we can do it with three observations. His compilation of solutions to nine illustrative surveying problems became the *Sea Island Mathematical Manual*. The mountain on the sea island is the first problem; the pine tree is the second. [**1**] and [**24**] include complete translations with commentary.

Here is the sea island problem:

> For looking at a sea island, erect two poles of the same height, 30 *chi*, the distance between the front and rear pole being 6000 *chi*. Assume that the rear pole is aligned with the front pole. Move away 738 *chi* from the front pole and observe the peak of the island from ground level; it is seen that the tip of the front pole coincides with the peak. Move backward 762 *chi* from the rear pole and observe the peak from ground level again; the tip of the rear pole also coincides with the peak. What is the height of the island and how far is it from the front pole?
>
> Answer: The height of the island is 7530 *chi*. It is 184500 *chi* from the front pole. [**24**, p. 20]

The extant version of the *Sea Island Manual* contains only the problems, answers, and recipes for obtaining the answers, exactly as in the *Nine Chapters*. Liu Hui also gave proofs for the correctness of his methods, but these proofs and the accompanying diagrams were not preserved, and the best we can do is offer plausible reconstructions. Using the notation of FIGURE 5, Liu's method for solution corresponds to the formulas

$$h = x + b = \frac{bd}{a_1 - a_2} + b, \qquad y = \frac{a_2 d}{a_1 - a_2}.$$

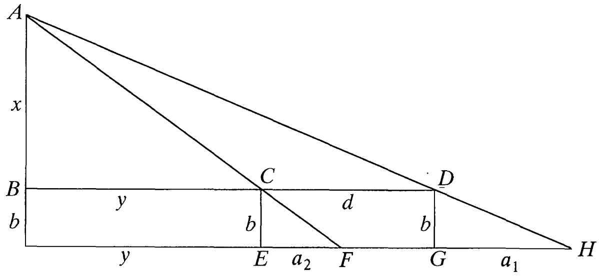


FIGURE 5
The height of a sea island.

We must obtain these formulas using only similar right triangles, since there was no concept of angle, much less any trigonometry, in ancient Chinese mathematics, nor was there any use of similar triangles other than right triangles. Here is one method. Since $\Delta ABD \sim \Delta DGH$,

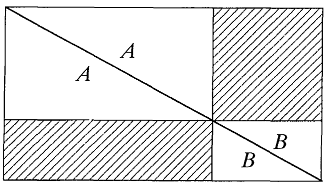
$$\frac{x}{y+d} = \frac{b}{a_1}, \quad \text{so } xa_1 = by + bd. \tag{1}$$

Since $\Delta ABC \sim \Delta CEF$,

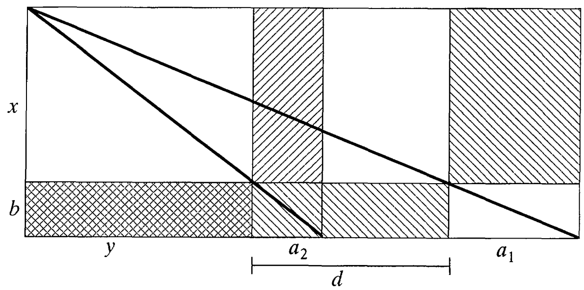
$$\frac{x}{y} = \frac{b}{a_2}, \quad \text{so } xa_2 = by. \tag{2}$$

Subtracting these equations gives $x(a_1 - a_2) = bd$, which leads to the expression for the height, and then substitution gives the distance.

Swetz **[24]** gives a very plausible alternate derivation which avoids the use of similar triangles completely. It is based on a lemma about rectangles which is illustrated in FIGURE 6a: if we divide a rectangle into four smaller rectangles at any point on its



(*a*) A rectangular lemma.



(*b*) A rectangular proof.

FIGURE 6

diagonal, then the two rectangles shaded in the figure must have the same area. This follows from a dissection argument. The diagonal divides the rectangle into two congruent triangles. From these triangles, subtracting the congruent triangles labeled A and B yields the given rectangles. If we apply this result twice to FIGURE 6b, the equal \\\ rectangles give equation (1), and the equal /// rectangles give equation (2). This method is also discussed in [**9**].

The *Sea Island Manual* was certainly not the deepest mathematics which Liu Hui did, but it probably had the greatest immediate impact. Recall that the kingdom of Wei was continually at war during the time of Liu's work. Surveying was important for maps which supported war, as well as the administrative bureaucracy. Needham reports that the Wei general Deng Ai always "estimated the heights and distances, measuring by finger breadths before drawing a plan of the place and fixing the position of his camp." [**24**, p. 15] There is an interesting parallel in the West. Swetz notes that Greek armies had a specific reason for wanting to calculate unknown height at an inaccessible distance, quoting Heron of Alexandria:

> How many times in the attack of a stronghold have we arrived at the foot of the ramparts and found that we made our ladders and other necessary implements for the assault too short, and have consequently been defeated simply for not knowing how to use the Dioptera for measuring the heights of walls; such heights have to be measured out of the range of enemy missiles. [**24**, p. 28]

The Calculation of π

Recall that problem 1.32 of the *Nine Chapters* gave the correct formula for the area of a circle, but used a value of three for π. Liu points out that for a circle of radius one, the area of a regular dodecagon inscribed in the circle is three, so the area of the circle must be greater than three. He then proceeds to estimate the area of the circle more exactly by calculating the areas of inscribed $3 \cdot 2^n$-gons as follows. In a circle of radius r, let c_n be the length of the side of an inscribed n-gon, a_n be the length of the perpendicular from the center of the circle to the side of the n-gon, and S_n be the area of the n-gon. See FIGURE 7. Then we can calculate inductively

$$\begin{aligned} c_6 &= r, \\ a_n &= \sqrt{r^2 - (c_n/2)^2}, \\ c_{2n} &= \sqrt{(c_n/2)^2 + (r - a_n)^2}, \\ S_{2n} &= \frac{1}{2} n r c_n. \end{aligned}$$

The last formula is clever, and follows from noticing that each of the $2n$ triangles making up the $2n$-gon can be thought of as having base r and height $c_n/2$. Moreover, FIGURE 7 shows that the area S of the circle satisfies

$$S_{2n} < S < S_n + 2(S_{2n} - S_n) = 2S_{2n} - S_n.$$

Liu considers what happens when we take n larger and larger: "the finer one cuts, the smaller the leftover; cut after cut until no more cut is possible; then it coincides with the circle and there is no leftover." [**20**, p. 347] As n gets large, S_{2n} approaches the area of the circle and nc_n approaches the circumference, so we have justified the *Nine Chapters* claim that the area of a circle is one-half the product of its radius and circumference.

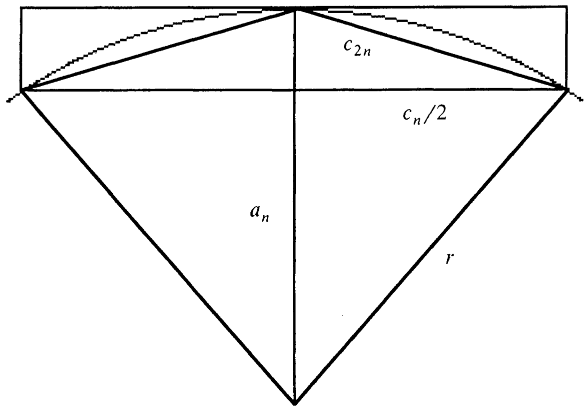


FIGURE 7
The calculation of π.

Taking $r = 10$, Liu Hui carries out the calculations, keeping 6-place accuracy, up to $n = 96$, hence approximating the circle by a 192-gon. He concludes that

$$3.1410 < \pi < 3.1427,$$

and suggests that for practical calculations it should be enough to use $\pi \approx 3.14$. Either Liu or some interpolating later commentator carried the computation as far as $n = 1536$ and obtained the approximation $\pi = 3.1416$. See **[13]** and **[28]** for treatments of the intricacies of this kind of calculation. **[13]** gives a translation of Liu Hui's text.

If we compare this treatment to Archimedes' in *Measurement of a Circle*, the similarities are striking, although the differences are also interesting. Archimedes, of course, included a formal proof by the method of exhaustion required by the conventions of Greek geometry. However, the subdivision method and the inductive calculation are essentially the same. Archimedes obtained his upper bound by considering circumscribed polygons, instead of Liu's clever method of using only inscribed polygons. Archimedes used 96-gons to obtain his famous estimate

$$3\frac{10}{71} < \pi < 3\frac{1}{7}, \quad \text{or} \quad 3.1409 < \pi < 3.1428.$$

Two centuries later Zu Chongzhi (429–500 A.D.) carried Liu Hui's approach farther. Using a polygon of 24576 sides, Zu obtained the bounds $3.1415926 < \pi < 3.1415927$. See **[13]** and, for a different view, **[28]**. In addition, Zu recommended two rational approximations for π: Archimedes' value of $22/7$, and the remarkably accurate $355/113 \approx 3.1415929$.

Zu's method for arriving at his rational approximation $\frac{355}{113}$ for π is not known. One line of reasoning would be to start with Zu's value of 3.1415926 and the approximation $\frac{22}{7} = 3\frac{1}{7} \approx 3.1428571$, which is slightly too large, and ask for a fraction which, when added to 3, would give a better approximation than $\frac{1}{7}$ does. It is easy to see that the fractions we should check are those of the form $\frac{k}{7k+1}$. We then try to find k so that

$$\frac{1}{7} - \frac{k}{7k+1} \approx .1428571 - .1415926 = .0012645,$$

$$\frac{1}{49k+7} \approx .0012645, \qquad 49k + 7 \approx 791.$$

The solution $k = 16$ gives the rational approximation $3\frac{16}{113} = \frac{355}{113}$. For another possible approach, see **[17]**.

Zu Chongzhi's approximation of π was not bettered until al-Kashi of Samarkand computed π to 14 decimal places in the early 15th century. The rational approximation 355/113 was not discovered in Europe until the late 16th century.

The Volume of Pyramids

Chapter Five of the *Nine Chapters* gives correct formulas for the volumes of a number of pyramidal solids. For example, the volume of the *chu-tung*, a truncated rectangular pyramid illustrated in FIGURE 11, is correctly given as

$$\frac{h}{6}(2ab + ad + bc + 2cd).$$

Did you know that formula? From it follows the volume of a rectangular pyramid (put $c = d = 0$), a truncated square pyramid (put $a = b$, $c = d$), and a rectangular wedge (put $d = 0$).

Liu Hui gives justifications for these formulas based on dissection arguments and a remarkable limit argument. I will mostly follow the translation and discussion in **[30]**. Liu's argument uses three special solids: a *qiandu*, which is a triangular prism, a *yangma*, which is a rectangular pyramid whose vertex is above one corner of its base, and a *bienao*, which is a tetrahedron with three successive perpendicular edges. See FIGURES 8, 9, and 10.

Liu starts with the case of a cube, which he dissects into three congruent *yangma*, to conclude that the volume of a regular *yangma* is 1/3 the volume of the cube. See FIGURE 8. Since a *yangma* and a *bienao* fit together to make a *qiandu*, which is 1/2 of the cube, the volume of the *bienao* must be 1/6 the volume of the cube. Alternatively, we could get this result by dissecting the *yangma* into two congruent *bienao*.

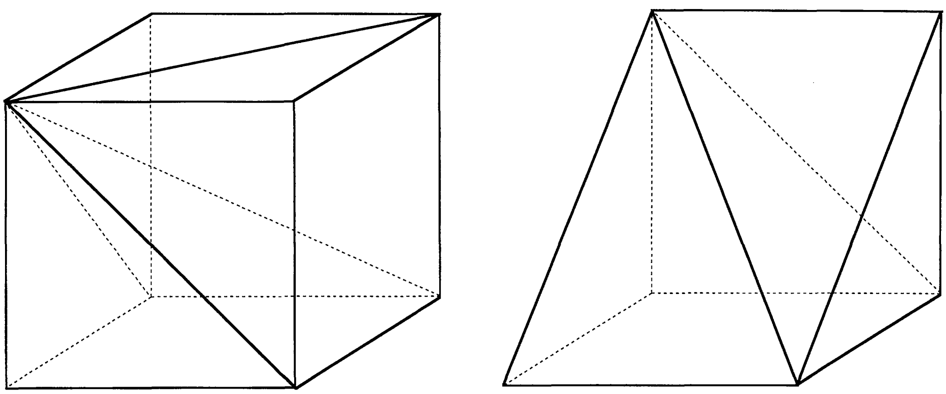


FIGURE 8
Dissecting a cube and a *qiandu*.

Now suppose that instead of a cube, we start with an $a \times b \times c$ rectangular box. We can still dissect it into three *yangma*, but now these *yangma* will have 3 different shapes, so it is not clear that their volumes are equal. We can also dissect a *yangma* into two *bienao*, or assemble a *bienao* and a *yangma* to make a *qiandu*, but again, the *bienao* have 3 different shapes, and it is not clear that their volumes are equal.

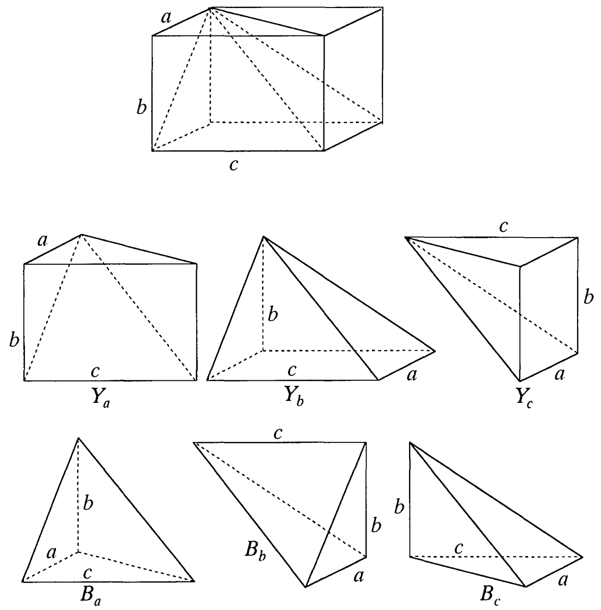


FIGURE 9
Three types of *yangma* and *bienao.*

Using the notation in FIGURE 9, what the dissections do show is that

$$Y_a + Y_b + Y_c = abc$$
$$Y_a + B_a = abc/2 \qquad Y_a = B_b + B_c$$
$$Y_b + B_b = abc/2 \qquad Y_b = B_a + B_c$$
$$Y_c + B_c = abc/2 \qquad Y_c = B_a + B_b.$$

However, this does not give enough information to evaluate the volumes.

Liu proceeds to prove that $Y_b = 2B_b$ (and similarly $Y_a = 2B_a$, $Y_c = 2B_c$), which does allow us to conclude that the volume of each *yangma* is $abc/3$ and that of each *bienao* is $abc/6$. His method is shown in FIGURE 10. Dissect Y_b at the midpoints of its sides into a rectangular box, 2 *qiandu*, and two half-size copies of Y_b (call them Y_b'). Similarly, dissect B_b into 2 *qiandu* and 2 half-size copies of B_b (call them B_b'). Since the box and 2 *qiandu* have twice the volume of 2 *qiandu*, we only need to show that

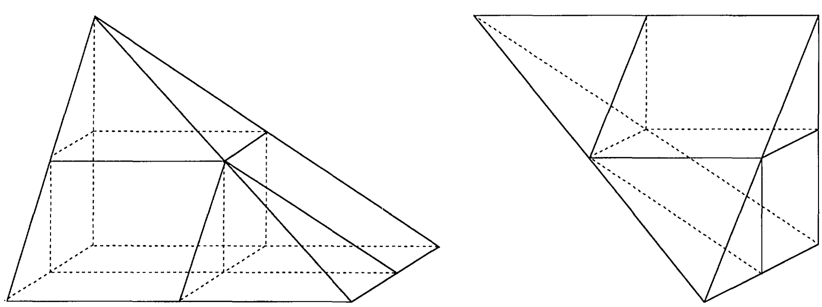


FIGURE 10
Dissecting a *yangma* and a *bienao*.

$Y_b' = 2B_b'$. Liu notes that these new figures together have $1/4$ the volume of the original figures, since the two small *yangma* and *bienao* fit together to form two *qiandu* whose total volume is $abc/8$. Repeat the dissection on each of the new figures, and continue. At each stage the volume we have not yet accounted for is $1/4$ that of the previous stage. Liu expresses what happens in the limit as follows:

> The smaller they are halved, the finer are the remaining dimensions. The extreme of fineness is called minute. That which is minute is without form. When it is explained in this way, why concern oneself with the remainder? [**30**, p. 173]

This is not a modern limit argument, of course. Liu seems to be saying that if we cut the figures into smaller and smaller pieces, we will come to a point where the pieces are so small that they no longer have form or volume. (The terms translated as 'minute' and 'form' are philosophical terms from the *Tao Te Ching*.) Still, we recognize the limit idea, and the recursive dissection argument has a delightful elegance. For some of the philosophical issues, see [**7**], [**16**], and [**30**]. For a comparison to the Greek proof in Euclid's *Elements*, see [**4**].

Knowing the volume of a *yangma*, we can now derive the volumes of the other solids by dissection. For example, let's verify the formula for the volume of the *chu-tung*. Dissect it as in FIGURE 11 into a box L, four *qiandu* of two different shapes Q_a and Q_b, and four *yangma* Y. If we do this to six copies of the *chu-tung*, we have

$$6L + 12Q_a + 12Q_b + 24Y.$$

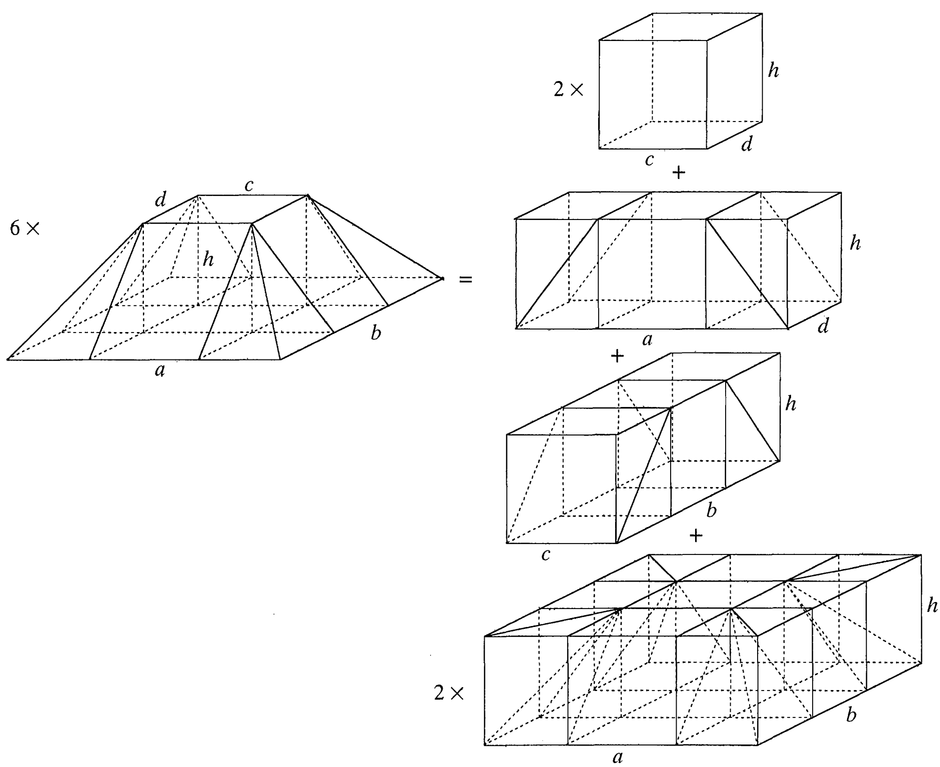


FIGURE 11
The volume of a *chu-tung*.

Now reassemble these, as in FIGURE 12, into

two boxes of volume hcd: $2L$
one box of volume had: $L+4Q_b$
one box of volume hbc: $L+4Q_a$
two boxes of volume hab: $2L+8Q_a+8Q_b+24Y$.

Notice that for the last step we need to replace some of the Y_h *yangma* with *yangma* of other shapes, but this is allowable since we have shown that these *yangma* all have the same volume.

Finally, Liu derives the volume of a cone from the volume of a square pyramid, and the volume of a truncated cone from the volume of a truncated square pyramid, by using what we know as "Cavalieri's principle," after Bonaventura Cavalieri (1598–1647). We can state this principle as follows:

> The volumes of two solids of the same height are equal if their planar cross-sections at equal heights always have equal areas; if the areas of the planar cross-sections at equal heights always have the same ratio, then the volumes of the solids also have this ratio.

Liu inscribes the truncated cone, for example, in a truncated square pyramid of the same height, and then says that since each cross-section consists of a circle inscribed in a square, the ratio of the volumes of the truncated cone to the truncated pyramid must be in the same ratio as the area of a circle to its circumscribed square, i.e., $\pi/4$ [**7**].

The Volume of a Sphere

Recall that problem 4.24 of the *Nine Chapters* gave the volume of a sphere as $\frac{9}{16}d^3$. Liu points out that this is incorrect, even using the inaccurate value of 3 for π. He explains the error as follows. Let a cylinder be inscribed in a cube of side d, and consider the cross-section of this figure by any plane perpendicular to the axis of the cylinder. The plane will cut the cylinder in a circle of diameter d, inscribed in a square of side d. The ratio of these areas is $\pi/4$. Since this is true for each cross-section, the same ratio must hold for the volumes, so that the volume of the cylinder is $\frac{\pi}{4}d^3$. Now consider the sphere of diameter d inscribed in the cylinder. If we assume, incorrectly, that the ratio of the volume of the sphere to the volume of the cylinder is also $\pi/4$, then we get that the volume of the sphere is $\frac{\pi^2}{16}d^3$, which is the *Nine Chapters* result (using $\pi=3$).

How do we know that the ratio of the volumes of the sphere and cylinder cannot be $\pi/4$? Liu's ingenious argument is as follows. Inscribe a second cylinder in the cube, with axis orthogonal to that of the first cylinder, and consider the intersection of these two cylinders. Liu called this intersection a "double box-lid." See FIGURE 12. Since the sphere is contained in both cylinders, it is contained in the box-lid. Moreover, consider any cross-section of this figure by a plane perpendicular to the axis of the box-lid. The cross-section of the sphere will be a circle, inscribed in the square which is the cross-section of the box-lid, so again the ratio of the areas is $\pi/4$, and since this is true for all cross-sections, the ratio of the volumes of the sphere and the box-lid must also be $\pi/4$. Now the box-lid is certainly smaller than the original cylinder, so the ratio of the volumes of the sphere and the cylinder must be strictly less than $\pi/4$.

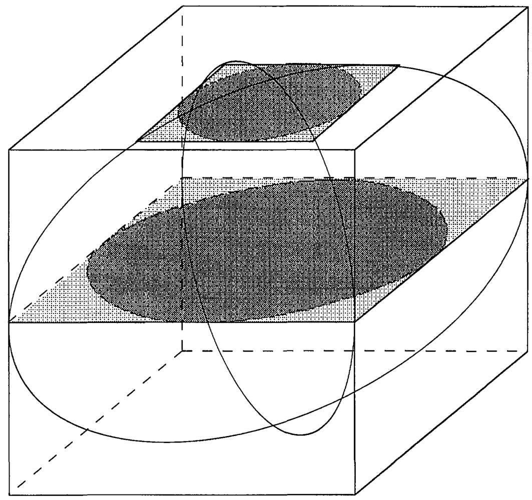


FIGURE 12
Cross sections of a sphere in a double box-lid in a cube.

This lovely argument using Cavalieri's principle shows that the *Nine Chapters* formula is wrong, but in order to use it to find the correct volume of the sphere, we would need to be able to find the volume of the double box-lid. Liu tried to do this, but could not. He recorded his failure in a poem, translated by D. B. Wagner as "The Geometer's Frustration:"

> Look inside the cube
> And outside the box-lid;
> Though the diminution increases,
> It doesn't quite fit.
>
> The marriage preparations are complete;
> But square and circle wrangle,
> Thick and thin make treacherous plots,
> They are incompatible.
>
> I wish to give my humble reflections,
> But fear that I will miss the correct principle;
> I dare to let the doubtful points stand,
> Waiting for one who can expound them. [**29**, p. 72]

The wait turned out to be two centuries, and the person Liu waited for was Zu Gengzhi, the son of Zu Chongzhi. Stories associated with Zu Gengzhi are reminiscent of those told about Archimedes and many mathematicians since then. For instance, "he studied so hard when he was still very young that he did not even notice when it thundered; when he was thinking about problems while walking he bumped into people." [**15**, p. 82]

Zu Gengzhi argues as follows. Consider one eighth of the double box-lid inscribed in the cube of side $r = d/2$. See FIGURE 13. If a plane is passed through this figure at height h, it intersects the cube in a square of side r, and the box-lid in a square of side s. By the *gou-gu* theorem, $r^2 - s^2 = h^2$. Hence the area of the gnomon *outside* the box-lid is h^2.

Now Zu Gengzhi considers another solid of height r whose cross-section at height h is h^2: an inverted *yangma* cut from a cube of side r. See FIGURE 13. The part of the cube outside the box-lid, and this *yangma*, have all their corresponding cross-sections of the same area. Zu then states his version of Cavalieri's principle in verse:

> If volumes are constructed of piled up blocks [areas],
> And corresponding areas are equal,
> Then the volumes cannot be unequal. [**29**, p. 75]

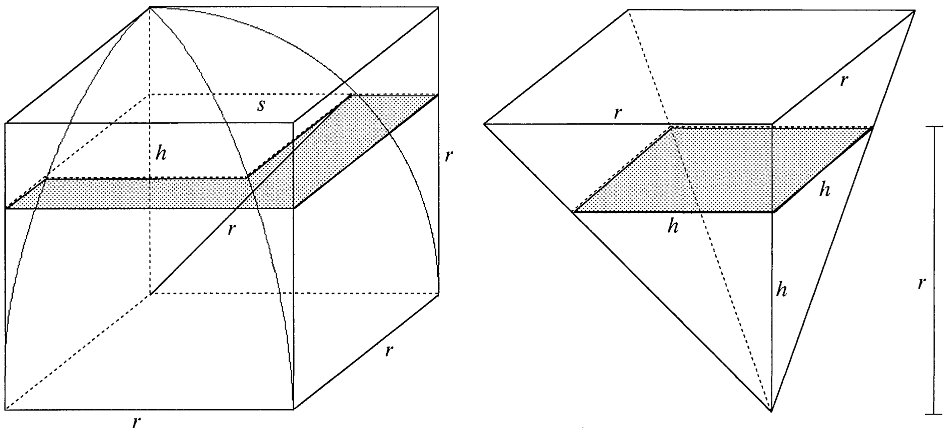


FIGURE 13
The volume outside a box-lid is Cavalieri-equivalent to a *yangma*.

Since the volume of the *yangma* is $\frac{1}{3}r^3$, and the volume outside the box-lid must be the same, the volume inside the box-lid must be $\frac{2}{3}r^3$. Putting the eight pieces together, we get that the volume of the complete double box-lid must be two-thirds of the cube containing it, $\frac{2}{3}d^3$. Remembering Liu Hui's result that the sphere takes up $\pi/4$ of the double box-lid, we finally get the correct formula for the volume of a sphere of diameter d:

$$V = \frac{\pi}{4}\frac{2}{3}d^3 = \frac{\pi}{6}d^3.$$

Following Liu, Zu ends his discussion with a poem, "The Geometer's Triumph:"

> The proportions are extremely precise,
> And my heart shines.
> Chang Heng copied the ancient,
> Smiling on posterity;
> Liu Hui followed the ancient,
> Having no time to revise it.
> Now what is so difficult about it?
> One need only think. [**29**, pp. 76–77]

One could argue that Liu Hui did not use the full power of Cavalieri's principle, since he only applied it to the situation of one figure inside another, where the cross-sections were circles inscribed in squares. But certainly Zu Gengzhi gave a clear statement of the principle and used its power more than a millennium before Cavalieri [**14**].

There was another precursor, of course. Archimedes had calculated the volume of a sphere, and in Proposition 15 of *The Method*, he calculated the volume of the perpendicular intersection of two cylinders of the same radius. The argument for Proposition 15 is in the part of *The Method* which has not survived, but it is not difficult to reconstruct the reasoning from other demonstrations earlier in the book. Archimedes thought of volumes as made up of planar slices and balanced them on a lever against the slices of other volumes. It is an extension of Cavalieri's principle. For a general discussion of the use of versions of Cavalieri's principle in Greek geometry, see **[10]**.

Conclusion

After the theoretical phase of Chinese mathematics in the 3rd through 5th centuries, represented by Liu Hui, Zu Chongzhi, and Zu Gengzhi, proofs and justifications began to be less important. Although the work of Liu Hui was still taught in the official School for the Sons of the State, instruction began to emphasize rote learning of methods rather than justifications. Liu's diagrams from the commentary on the *Nine Chapters* and arguments from the *Sea Island Manual*, and Zu Chongzhi's work, were lost. The next, brief flowering of creative mathematics in China did not happen until the 13th century, with mathematicians like Qin Jiushao, Li Zhi, Zhu Shijie, and Yang Hui. After the thirteenth century, Chinese mathematics declined again until the period of contact with the West.

It is interesting to speculate why Chinese mathematics, with such a powerful calculational base and such a strong theoretical start, did not develop a coherent, ongoing mathematical tradition. Martzloff **[17]** and Swetz **[25]** review a number of possible reasons: emphasis on practical applications, rote learning, and reverence for established ideas which stifled creativity, uneven state support, and low social status accorded to mathematicians compared to scholars in the humanities.

Nevertheless, the remarkable achievements of Chinese mathematics in its first golden age are worthy of our interest and admiration.

Acknowledgment. I wish to thank the mathematics department of the University of Colorado at Boulder for their hospitality during the writing of this paper, and Victor Katz, Ranjan Roy, and Frank Swetz for suggestions which have improved its quality.

Note. **[8]** and **[21–27]** contain very accessible introductions to Chinese mathematics. **[15]** and **[17]** are comprehensive modern histories of Chinese mathematics which make extensive use of Chinese research. **[18]** and **[19]** are older histories which are still good reading.

REFERENCES

1. Ang Tian Se and Frank Swetz, A Chinese mathematical classic of the third century: The Sea Island Mathematical Manual of Liu Hui, *Historia Mathematica* 13 (1986) 99–117.
2. David Burton, *Burton's History of Mathematics: An Introduction*, Wm. C. Brown, Dubuque, IA, 1995.
3. Karine Chemla, Theoretical aspects of the Chinese algorithmic tradition (first to third century), *Historia Scientiarum* 42 (1991) 75–98.
4. J. N. Crossley and A. W. C. Lun, The logic of Liu Hui and Euclid as exemplified in their proofs of the volume of a pyramid, *Philosophy and the History of Science: A Taiwanese Journal*, 3 (1994) 11–27.
5. T. L. Heath, ed., *The Works of Archimedes*, Dover, New York, NY, 1912.
6. Ho Peng-Yoke, Liu Hui, *Biographical Dictionary of Mathematicians*, Charles Scribner's Sons, New York, NY, 1991.
7. Horng Wann Sheng, How did Liu Hui perceive the concept of infinity: a revisit, *Historia Scientiarum* 4 (1995) 207–222.

8. G. Joseph, *The Crest of the Peacock*, Penguin, 1991.
9. Victor Katz, *A History of Mathematics: An Introduction*, Harper Collins, 1993.
10. Wilbur Knorr, The method of indivisibles in ancient geometry, in R. Calinger, ed., *Vita Mathematica*, Mathematical Association of America, 1996.
11. Lam Lay Yong, Jiu Zhang Suanshu (Nine Chapters on the Mathematical Art): an overview, *Archive for History of Exact Sciences* 47 (1994) 1–51.
12. Lam Lay Yong, Hindu-Arabic and traditional Chinese arithmetic, *Chinese Science* 13 (1996) 35–54.
13. Lam Lay Yong and Ang Tian Se, Circle measurements in ancient China, *Historia Mathematica* 13 (1986) 325–340.
14. Lam Lay Yong and Shen Kangsheng, The Chinese concept of Cavalieri's principle and its applications, *Historia Mathematica* 12 (1985) 219–228.
15. Li Yan and Du Shiran, *Chinese Mathematics: A Concise History*, translated by J. Crossley and A. Lun, Oxford University Press, Oxford, UK, 1987.
16. Geoffrey Lloyd, Finite and infinite in Greece and China, *Chinese Science* 13 (1996) 11–34.
17. Jean-Claude Martzloff, *A History of Chinese Mathematics*, Springer-Verlag, New York, NY, 1997.
18. Yoshio Mikami, *The Development of Mathematics in China and Japan*, Chelsea, New York, NY, 1913.
19. Joseph Needham, *Science and Civilisation in China*, vol. 3: *Mathematics and the Sciences of the Heavens and the Earth*, Cambridge University Press, Cambridge, UK, 1959.
20. Siu Man-Keung, Proof and pedagogy in ancient China: examples from Liu Hui's commentary on Jiu Zhang Suan Shu, *Educational Studies in Mathematics* 24 (1993) 345–357.
21. *Frank Swetz, The amazing Chiu Chang Suan Shu, Mathematics Teacher* 65 (1972) 425–430. Reprinted in F. Swetz, ed., *From Five Fingers to Infinity*, Open Court, Chicago, IL, 1994.
22. Frank Swetz, The 'piling up of squares' in ancient China, *Mathematics Teacher* 70 (1975) 72–79. Reprinted in F. Swetz, ed., *From Five Fingers to Infinity*, Open Court, Chicago, IL, 1994.
23. Frank Swetz, The evolution of mathematics in ancient China, this MAGAZINE, 52 (1979) 10–19. Reprinted in F. Swetz, ed., *From Five Fingers to Infinity*, Open Court, Chicago, IL, 1994.
24. Frank Swetz, *The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China*, Pennsylvania State University Press, 1992.
25. Frank Swetz, Enigmas of Chinese mathematics, in R. Calinger, ed., *Vita Mathematica*, Mathematical Association of America, Washington, DC, 1996.
26. Frank Swetz and T. I. Kao, *Was Pythagoras Chinese? An Examination of Right Triangle Theory in Ancient China*, Pennsylvania University Press, 1977.
27. Robert Temple, *The Genius of China*, Simon and Schuster, New York, NY, 1986.
28. Alexei Volkov, Calculation of π in ancient China: from Liu Hui to Zu Chongzhi, *Historia Scientiarum* 4 (1994) 139–157.
29. D. B. Wagner, Liu Hui and Tsu Keng-chih on the volume of a sphere, *Chinese Science* 3 (1978) 59–79.
30. D. B. Wagner, An early Chinese evaluation of the volume of a pyramid: Liu Hui, third century A.D., *Historia Mathematica* 6 (1979) 164–188.
31. D. B. Wagner, A proof of the Pythagorean theorem by Liu Hui (third century A.D.), *Historia Mathematica* 12 (1985) 71–73. l