The Evolution of Mathematics in Ancient China

Early Chinese mathematical accomplishments reveal arithmetic and algebraic approaches based on sophisticated inductive knowledge.

FRANK SWETZ
The Pennsylvania State University
Middletown, PA 17057

A popular survey book on the development of mathematics has its text prefaced by the following remarks:

> Only a few ancient civilizations, Egypt, Babylonia, India and China, possessed what may be called the rudiments of mathematics. The history of mathematics and indeed the history of western civilization begins with what occurred in the first of these civilizations. The role of India will emerge later, whereas that of China may be ignored because it was not extensive and moreover has no influence on the subsequent development of mathematics.[1]

Even most contemporary works on the history of mathematics reinforce this impression, either by neglecting or depreciating Chinese contributions to the development of mathematics.[2] Whether by ignorance or design, such omissions limit the perspective one might obtain concerning both the evolution of mathematical ideas and the place of mathematics in early societies. In remedying this situation, western historians of mathematics may well take heed of Whittier's admonition:

> We lack but open eye and ear
> To find the Orient's marvels here.[3]

Language barriers may limit this quest for information; however, a search of English language sources will reveal that there are many "marvels" in Chinese mathematics to be considered.

Legend and Fact

The origins of mathematical activity in early China are clouded by mysticism and legend. Mythological Emperor Yü is credited with receiving a divine gift from a Lo river tortoise. The gift in the form of a diagram called the *Lo shu* is believed to contain the principles of Chinese mathematics, and pictures of Yü's reception of the *Lo shu* have adorned Chinese mathematics books for centuries. This fantasy in itself provides some valuable impressions about early Chinese science and mathematics. Yü was the patron of hydraulic engineers; his mission was to control the flood-prone waters of China and provide a safe setting in which a water-dependent civilization could flourish. The users of science and mathematics in China were initially involved with hydraulic engineering projects, the construction of dikes, canals, etc., and with the mundane tasks of logistically supporting such projects. A close inspection of the contents of the *Lo shu* reveals a number configuration (FIGURE 1) which would be known later in the West as a magic square. For Chinese soothsayers and geomancers from the Warring State period of Chinese history (403–221 B.C.) onward, this square, comprised of numbers, possessed real magical qualities because in it they saw a plan of universal harmony based on a cosmology predicated on the dualistic theory of the *Yin* and the *Yang*.[4]

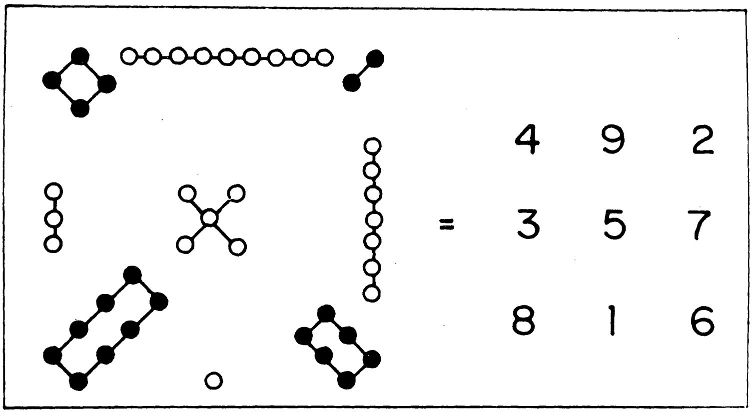


FIGURE 1.

When stripped of ritualistic significance, the principles used in constructing this first known magic square are quite simple and can best be described by use of diagrams as shown in FIGURE 2. The construction and manipulation of magic squares became an art in China even before the concept was known in the West.[5] Variations of the *Lo shu* technique were used in constructing magic squares of higher order with perhaps the most impressive square being that of order nine; see FIGURE 3.

```
                         1                    9
 7 4 1                 4   2                4   2               4 9 2
 8 5 2               7   5   3            3   5   7             3 5 7
 9 6 3                 8   6                8   6               8 1 6
                         9                    1
```

Construct a natural square. Distort it into a diamond. Exchange corner elements. Compress back into a square.

FIGURE 2.

1	10	19	28	37	46	55	64	73
2	11							·
3	12							·
4	13							·
5	14							·
6	15							·
7	16							·
8	17							·
9	18							81

(a)

55	28	1
64	37	10
73	46	19

(b)

28	73	10
19	37	55
64	1	46

(c)

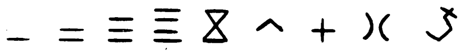
31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	(4)	49	72	(9)	54	65	(2)	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	(3)	48	68	(5)	50	70	(7)	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	(8)	53	64	(1)	46	69	(6)	51

(d)

Start with a natural square (a) then fold each row into a square (b) of order 3 (example using row 1) and apply the *Lo shu* technique (c). The nine resulting magic squares of order 3 (d) are then positionally ordered according to the correspondence of the central element in their bottom rows with the numbers of the *Lo shu*, i.e., 4,9,2; 3,5,7; 8,1,6.

FIGURE 3.

While the *Lo shu* provides some intriguing insights into early mathematical thinking, its significance in terms of potential scientific or technological achievement is negligible. Historically, the first true evidence of mathematical activity can be found in numeration symbols on oracle bones dated from the Shang dynasty (14th century B.C.). Their numerical inscriptions contain both tally and code symbols, are clearly decimal in their conception, and employ a positional value system. The Shang numerals for the numbers one through nine were:



By the time of the Han Dynasty (2nd century B.C.–4th century A.D.), the system had evolved into a codified notation that lent itself to computational algorithms carried out with a counting board and set of rods. The numerals and their computing-rod configurations are

1	2	3	4	5	6	7	8	9		
𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱	for coefficients of 10^{2n-2}	$n=1,2\ldots$
𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨	for coefficients of 10^{2n-1}	$n=1,2\ldots.$

Thus in this system 4716 would be represented as 𝍣 𝍯 𝍠 𝍮.[6] (Occasionally the symbol × was used as an alternative to 𝍭.)

Counting boards were divided into columns designating positional groupings by 10. The resulting facility with which the ancient computers could carry out algorithms attests to their full understanding of decimal numeration and computation. As an example, consider the counting board method of multiplying 2 three-digit numbers, as illustrated in FIGURE 4. The continual indexing of partial products to the right as one multiplies by smaller powers of ten testifies to a thorough understanding of decimal notation. In light of such evidence, it would seem that the Chinese were the first society to understand and efficiently utilize a decimal numeration system.[7] If one views a popular schematic of the evolution of our modern system of numeration (FIGURE 5) and places the Chinese system in the

Counting board

		2	4	6	(multiplier)
					(product)
		3	5	7	(multiplicand)

		2	4	6
7	1	4		
3	5	7		

			4	6
8	5	6	8	
	3	5	7	

				6	
8	7	8	2	2	(answer)
		3	5	7	

Accompanying rod computations

2×3=6
2×5= 10
70

2×7= 14
714
4×3= 12
834

4×5= 20
854
4×7= 28
8568
6×3= 18
8748

6×5= 30
8778
6×7= 42
87822

FIGURE 4.

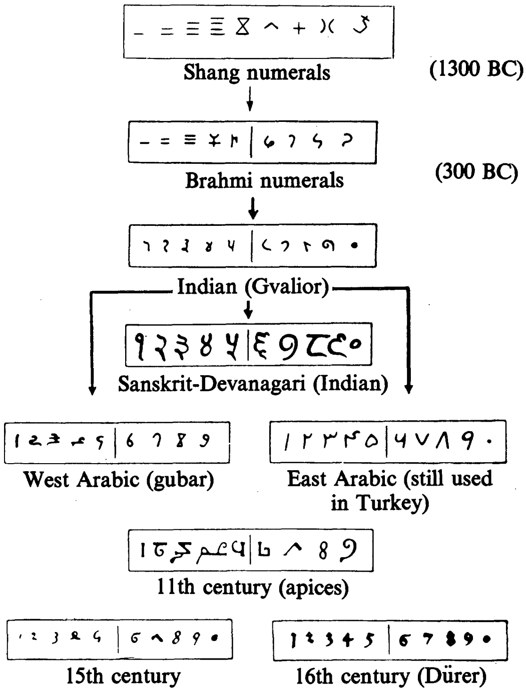


FIGURE 5.

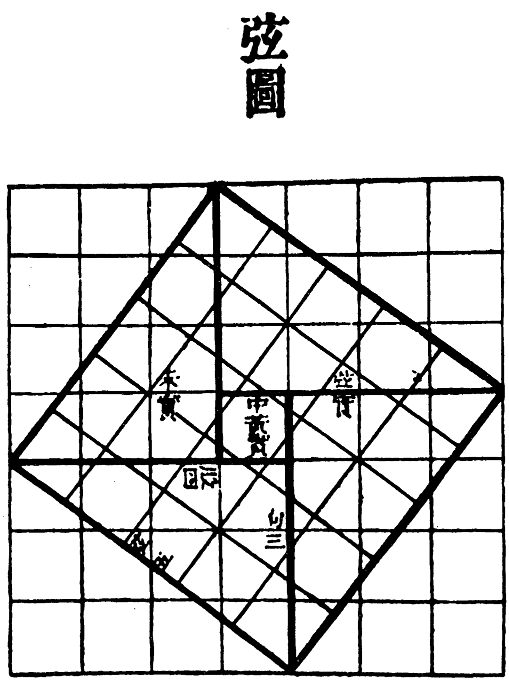


FIGURE 6.

appropriate chronological position, an interesting hypothesis arises, namely that the numeration system commonly used in the modern world had its origins 34 centuries ago in Shang China!

The Systematization of Early Chinese Mathematics

The oldest extant Chinese text containing formal mathematical theories is the *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven*, [*Chou pei suan ching*]. Its contents date before the third century B.C. and reveal that mathematicians of the time could perform basic operations with fractions according to modern principles employing the concept of common denominator. They were knowledgeable in the principles of an empirical geometry and made use of the "Pythagorean theorem." A diagram (see FIGURE 6) in the *Chou pei* presents the oldest known demonstration of the validity of this theorem. This diagram, called the *hsuan-thu* in Chinese, illustrates the arithmetic-geometric methodology that predominates in early Chinese mathematical thinking and shows how arithmetic and geometry could be merged to develop algebraic processes and procedures. If the oblique square of the *hsuan-thu* is dissected and the pieces rearranged so that two of the four congruent right triangles are joined with the remaining two to form two rectangles, then the resulting figure comprised of two rectangles and one small square have the same area as their parent square. Further, since the new configuration can also be viewed as being comprised of two squares whose sides are the legs of the right triangles, this figure demonstrates that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.[8] The processs involved in this intuitive, geometric approach to obtain algebraic results was called *chi-chü* or "the piling up of squares."[9]

The next historical text known to us is also a Han work of about the third century B.C. It is the *Nine Chapters on the Mathematical Art*, [*Chiu chang suan shu*], and its influence on oriental mathematics may be likened to that of Euclid's *Elements* on western mathematical thought. The *Chiu chang's* chapters bear such titles as surveying of land, consultations on engineering works, and impartial taxation, and confirm the impression that the Chinese mathematics of this period centered on the engineering and bureaucratic needs of the state. Two hundred and forty-six problem situations are considered, revealing in their contents the fact that the Chinese had accumulated a variety of formulas for determining the areas and volumes of basic geometric shapes. Linear equations in one unknown were solved by a rule of false position. Systems of equations in two or three unknowns were solved simultaneously by computing board techniques that are strikingly similar to modern matrix methods. While algebraists of the ancient world such as Diophantus or Brahmagupta used various criteria to distinguish between the variables in a linear equation,[10] the Chinese relied on the organizational proficiency of their counting board to assist them in this chore. Using a counting board to work a system of equations allowed the Chinese to easily distinguish between different variables.

Consider the following problem from the *Chiu chang* and the counting board approach to its solution.

> Of three classes of cereal plants, 3 bundles of the first; 2 of the second and 1 of the third will produce 39 *tou* of corn after threshing; 2 bundles of the first; 3 of the second and 1 of the third will produce 34 *tou*; while 1 of the first, 2 of the second and 3 of the third will produce 26 *tou*. Find the measure of corn contained in one bundle of each class.[11]
>
> (1 *tou* = 10.3 liters)

This problem would be set up on the counting board as:

1	2	3	1st class grain
2	3	2	2nd class grain
3	1	1	3rd class grain
26	34	39	Number of *tou*

Using familiar notation this matrix of numbers is equivalent to the set of equations

$$3x+2y+z=39$$
$$2x+3y+z=34$$
$$x+2y+3z=26$$

which are reduced in their tabular form by appropriate multiplications and subtraction to

$$\begin{aligned} 3x+2y+\ z &= 30 \\ 36y \quad &= 153 \\ 36z &= 99 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 36x \qquad &= 333 \\ 36y \quad &= 153 \\ 36z &= 99. \end{aligned}$$

Thus $x=333/36$, $y=153/36$ and $z=99/36$.

A companion problem from the *Chiu chang* involves payment for livestock and results in the system of simultaneous equations:

$$-2x+5y-13z=1000$$
$$3x-9y+3z=0$$
$$-5x+6y+8z=-600.$$

Rules provided for the solution treat the addition and subtraction of negative numbers in a modern fashion; however, procedures for the multiplication and division of negative numbers are not found in a Chinese work until the Sung dynasty (+1299). Negative numbers were represented in the computing scheme by the use of red rods, while black computing rods represented positive numbers. Zero was indicated by a blank space on the counting board. This evidence qualifies the Chinese as being the first society known to use negative numbers in mathematical calculations.

The *Chou pei* contains an accurate process of extracting square roots of numbers. The ancient Chinese did not consider root extraction a separate process of mathematics but rather merely a form

$$166536 \div 648$$

Counting board layout		Accompanying rod computations	Explanations
2	(quotient)	166500	200 is chosen
		$-120000 = (200\times 600)$	as the first
166536	(dividend)	46500	partial
		$-8000 = (200\times 40)$	quotient
648	(divisor)	38500	
		$-1600 = (200\times 8)$	
		36900	
25		36930	50 is chosen
		$-30000 = (50\times 600)$	as the second
36936		6930	partial
		$-2000 = (50\times 40)$	quotient
648		4930	
		$400 = (50\times 8)$	
		4530	
257		4536	7 is chosen
		$-4200 = (7\times 600)$	as the third
4536		336	partial
		$-280 = (7\times 40)$	quotient
648		56	
		$-56 = (7\times 8)$	
		0	process is finished

FIGURE 7.

of division.[12] Let us examine the algorithm for division and its square root variant. The division algorithm is illustrated in FIGURE 7 for the problem $166536 \div 648$. The Chinese technique of root extraction depends on the algebraic proposition

$$\begin{aligned}(a+b+c)^2 &= a^2+2ab+b^2+2(a+b)c+c^2\\ &= a^2+(2a+b)b+(2[a+b]+c)c\end{aligned}$$

which is geometrically substantiated by the diagram given in FIGURE 8. This proposition is incorporated directly into a form of division where $\sqrt{N} = a+b+c$. The counting board process for extracting the square root of 55225 is briefly outlined in FIGURE 9. Root extraction was not limited to three digit results, for the Chinese were able to continue the process to several decimal places as needed. Decimal fractions were known and used in China as far back as the 5th century B.C. Where a root was to be extracted to several decimal places, the computers achieved greater accuracy by use of the formulae $\sqrt[n]{m} = \sqrt[n]{m10^{kn}}\,/10^k$.[13] Cube root extraction was conceived on a similar geometric-algebraic basis and performed with equal facility.

Historians of mathematics often devote special consideration to the results obtained by ancient societies in determining a numerical value for π as they believe that the degree of accuracy achieved supplies a comparative measure for gauging the level of mathematical skill present in the society. On the basis of such comparisons, the ancient Chinese were far superior to their contemporaries in computational mathematical ability. Aided by a number system that included the decimalization of fractions and the possession of an accurate root extraction process the Chinese had obtained by the first century a value of π of 3.15147. The scholar Liu Hui in a third century commentary on the *Chiu chang* employed a "cutting of the circle method"—determining the area of a circle with known radius by polygonal approximations—to determine π as 3.141024. A successor, Tsu Chung-chih, refined the method in the fifth century to derive the value of π as 355/113 or 3.1415929.[14] This accuracy was not to be arrived at in Europe until the 16th century.

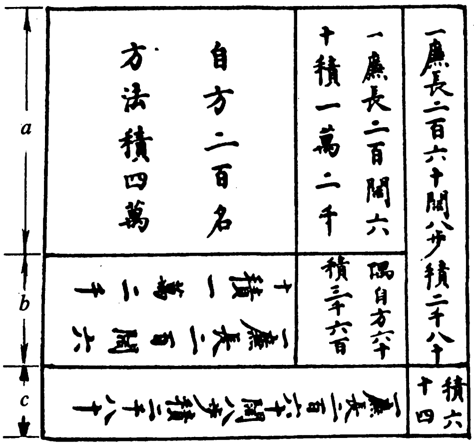


FIGURE 8.

A geometric "proof" (FIGURE 8) of the algebraic proposition (see p. 14) which justifies the calculations (FIGURE 9) leading to $\sqrt{55225}=235$. The 1 in the upper box represents an indexing rod that determines the decimal value of the divisors used. At the beginning of the process, it is moved to the left in jumps of two decimal places until it establishes the largest power of ten that can be divided into the designated number. After each successful division, the rod is indexed two positional places to the right.

Algebraic Significance	Numerical entries on board
N 1	55225 1
a $N-a^2$ $a\times 10000$ 10000	2 15225 20000 10000
$a+b$ $N-a^2$ $(2a+b)b\times 100$ $(2a+b)\times 100$ 100	23 15225 12900 4300 100
$a+b$ $N-[a^2+(2a+b)b]$ $(2a+b)\times 100$ 100	23 2325 4300 100
$a+b+c$ $N-(a+b)^2-[2a(a+b)+c]c$ $2(a+b)+c$	235 0 465

FIGURE 9.

Trends in Chinese Algebraic Thought

While the Chinese computational ability was indeed impressive for the times, their greatest accomplishments and contributions to the history of mathematics lay in algebra. During the Han period, the square and cube root extraction processes were being built upon to obtain methods for solving quadratic and other higher order numerical equations. The strategy for extending the square root process to solve quadratic equations was based on the following line of reasoning. If $x^2=289$, 10 would be chosen as a first entry approximation to the root, then

$$289-(10)^2=189.$$

Let the second entry of the root be represented by y; thus, $x=10+y$ or $(10+y)^2=289$ which, if expanded, gives the quadratic equation $y^2+20y-189=0$. By proceeding to find the second entry of the square root of 289, 7, we obtain the positive root for the quadratic $y^2+20y-189=0$.[15]

By the time of Sung Dynasty in the 13th century, mathematicians were applying their craft to solve such challenging problems as:

> This is a round town of which we do not know the circumference or diameter. There are four gates (in the wall). Three *li* from the northern (gate) is a high tree. When we go outside of the southern gate and turn east, we must walk 9 *li* before we see the tree. Find the circumference and the diameter of the town. (1 *li* = .644 kilometers)

If the diameter of the town is allowed to be represented by x^2, the distance of the tree from the northern gate, a, and the distance walked eastward, b, the following equation results.

$$x^{10}+5ax^8+8a^2x^6-4a(b^2-a^2)x^4-16a^2b^2x^2-16a^3b^2=0.$$

For the particular case cited above, the equation becomes

$$x^{10}+15x^8+72x^6-864x^4-11{,}664x^2-34{,}992=0.$$

Sung algebraists found the diameter of the town to be 9 *li*.[16]

The earliest recorded instance of work with indeterminate equations in China can be found in a problem situation of the *Chiu chang* where a system of four equations in five unknowns results.[17] A particular solution is supplied. A problem in the third century *Mathematical Classic of Sun Tzu*, [*Sun Tzu suan ching*,] concerns linear congruence and supplies a truer example of indeterminate analysis.

> We have things of which we do not know the number; if we count by threes, the remainder is 2; if we count by fives, the remainder is 3; if we count by sevens, the remainder is 2. How many things are there?[18]

In modern form, the problem would be represented as:

$$N \equiv 2 \pmod 3 \equiv 3 \pmod 5 \equiv 2 \pmod 7.$$

Sun's solution is given by the expression

$$70\times 2+21\times 3+15\times 2-105\times 2=23$$

which when analysed gives us the first application of the Chinese Remainder Theorem.

If $m_1,\ldots,m_k$ are relatively prime in pairs, there exist integers x for which simultaneously $x\equiv a_1 \pmod{m_1},\ldots,x\equiv a_k \pmod{m_k}$. All such integers x are congruent modulo $m=m_1m_2\ldots m_k$. The existence of the Chinese Remainder Theorem was communicated to the west by Alexander Wylie, an English translator and mathematician in the employ of the nineteenth century Chinese court. Wylie recorded his findings in a series of articles, "Jottings on the Science of the Chinese; Arithmetic" which appeared in the *North China Herald* (Aug.–Nov.) 1852. The validity of the theorem was questioned until it was recognized as a variant of a formula developed by Gauss.[19]

Perhaps the most famous Chinese problem in indeterminate analysis, in the sense of its transmission to other societies, was the problem of the "hundred fowls" (ca 468).

> A cock is worth 5 *ch'ien*, a hen 3 *ch'ien*, and 3 chicks 1 *ch'ien*. With 100 ch'ien we buy 100 fowls. How many cocks, hens, and chicks are there?
>
> (*ch'ien*, a small copper coin)

The development of algebra reached its peak during the later part of the Sung and the early part of the following Yuan dynasty (13th and 14th centuries). Work with indeterminate equations and higher order numerical equations was perfected. Solutions of systems of equations were found by using methods that approximate an application of determinants, but it wasn't until 1683 that the Japanese Seki Kowa, building upon Chinese theories, developed a true concept of determinants.

Work with higher numerical equations is facilitated by a knowledge of the binomial theorem. The testimony of the *Chiu chang* indicates that its early authors were familiar with the binomial expansion $(a+b)^3$, but Chinese knowledge of this theorem is truly confirmed by a diagram (FIGURE 10) appearing in the 13th century text *Detailed Analysis of the Mathematical Rules in the Nine Chapters*. [*Hsiang chieh chiu chang suan fa*.]. It seems that "Pascal's Triangle" was known in China long before Pascal was even born.

While mathematical activity continued in the post-Sung period, its contributions were minor as compared with those that had come before. By the time of the Ming emperors in the 17th century, western mathematical influence was finding its way into China and the period of indigenous mathematical accomplishment had come to an end.

Conclusions

Thus, if comparisons must be made among the societies of the pre-Christian world, the quality of China's mathematical accomplishments stands in contention with those of Greece and Babylonia, and

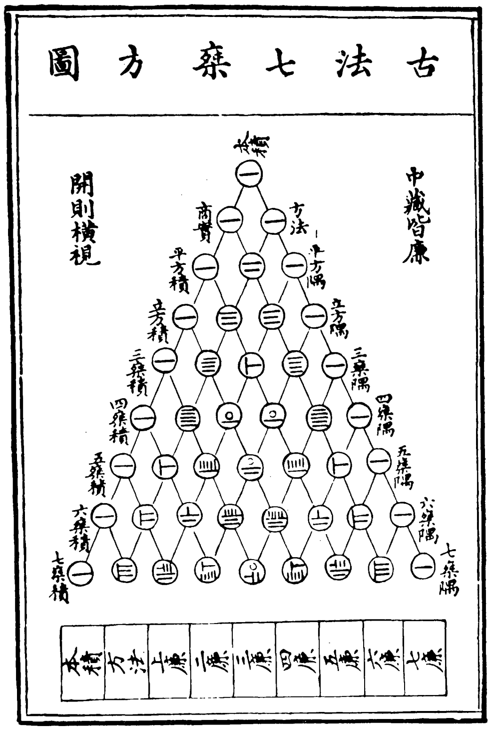


FIGURE 10.

during the period designated in the West as pre-Renaissance, the sequence and scope of mathematical concepts and techniques originating in China far exceeds that of any other contemporary society. The impact of this knowledge on the subsequent development of western mathematical thought is an issue that should not be ignored and can only be resolved by further research. In part, such research will have to explore the strength and vitality of Arabic-Hindu avenues of transmission of Chinese knowledge westward. The fact that western mathematical traditions are ostensibly based on the logico-deductive foundations of early Greek thought should not detract from considering the merits of the inductively-conceived mathematics of the Chinese. After all, deductive systemization is a luxury afforded only after inductive and empirical experimentation has established a foundation from which theoretical considerations can proceed. Mathematics, in its primary state, is a tool for societal survival; once that survival is assured, the discipline can then become more of an intellectual and aesthetic pursuit. Unfortunately, this second stage of mathematical development never occurred in China. This phenomenon—the fact that mathematics in China, although developed to a high art, was never elevated further to the status of an abstract deductive science—is yet another fascinating aspect of Chinese mathematics waiting to be explained.

Notes

1. Morris Kline, *Mathematics*: *A Cultural Approach* (Reading, Mass.: Addison-Wesley Publishing Co. 1962) p. 12.
2. In his 712 page *A History of Mathematics* (New York: John Wiley & Sons Inc., 1968) Carl Boyer devotes 12 pages to Chinese contributions; the latest revised edition of Howard Eves, *An Introduction to the History of Mathematics*, (New York: Holt, Rinehart and Winston, 1976) contains 6 pages on the history of Chinese mathematics. The contents of these pages are based on information given in an article by D. J. Struik, "On Ancient Chinese Mathematics," *The Mathematics Teacher* (1963), 56: 424–432 and represent little of Eves' own research.
3. John Greenleaf Whittier, "The Chapel of the Hermits."
4. Under this system, the universe is ruled by Heaven through means of a process called the *Tao* ("the Universal way"). Heaven acting through the Tao expresses itself in the interaction of two primal forces, the *Yin* and the

Yang. The *Yang*, or male force, was a source of heat, light and dynamic vitality and was associated with the sun; in contrast, the *Yin*, or female force, flourished in darkness, cold and quiet inactivity and was associated with the moon. In conjunction, these two forces influenced all things and were present individually or together in all physical objects and situations. In the case of numbers, odd numbers were *Yang* and even, *Yin*. For a harmonious state of being to exist, *Yin-Yang* forces had to be balanced.

5. For a fuller discussion of Chinese magic squares, see Schyler Camman, "Old Chinese Magic Squares", *Sinologica* (1962), 7 : 14–53; Frank Swetz, "Mysticism and Magic in the Number Squares of Old China," *The Mathematics Teacher* (January, 1978), 71: 50–56.
6. The evolution of counting rod numerals continued for about 3000 years in China, i.e., 14th century BC–13th century AD. For a discussion of this process, see Joseph Needham, *Science and Civilization in China* (Cambridge: Cambridge University Press, 1955) vol. 3. pp. 5–17.
7. A strong case for this theory has been made by Wang Ling, "The Chinese Origin of the Decimal Place—Value System in the Notation of Numbers". Communication to the 23rd International Congress of Orientalists, Cambridge, 1954.
8. Although a 3, 4, 5 right triangle is used in the demonstration, the Chinese generalized their conclusion for all right triangles. The 3, 4, 5 triangle was merely a didactical aid.
9. See Frank Swetz, "The 'Piling Up of Squares' in Ancient China," *The Mathematics Teacher* (1977), 70: 72–79.
10. Diophantus (275 AD) spoke of unknowns of the first number, second number, etc., whereas Brahmgupta (628 AD) used different colors in written computations to distinguish between variables.
11. *Chiu chang suan shu*, Fang Chheng (chapter 8), problem 1.
12. For a discussion of the Chinese ability at root extraction, see Wang Ling and Joseph Needham, "Horner's Method in Chinese Mathematics: Its Origins in the Root Extraction Procedures of the Han Dynasty", *T'oung Pao* (1955), 43: 345–88; Lam Lay Yong, "The Geometrical Basis of the Ancient Chinese Square-Root Method", *Isis* (Fall, 1970), pp. 92–101.
13. A lengthy discussion of the use of this formula in Europe is given in D. E. Smith, *History of Mathematics* (New York: Dover Publishing Co., 1958 reprint) vol. II, p. 236.
14. The evolution of π in China is traced out in Lee Kiong-Pong, "Development of π in China", *Bulletin of the Malaysian Mathematical Society* (1975), 6:40–47.
15. An actual computational procedure used in solving quadratics can be found in Ho Peng Yoke, "The Lost Problems of the Change Ch'iu-chien Sual Ching, a Fifth Century Chinese Mathematical Manual," *Oriens Extremus* (1965), 12.
16. For a detailed discussion of the solution of this problem see Ulrich Libbrecht, *Chinese Mathematics in the Thirteenth Century* (Cambridge, Mass.: The MIT Press, 1973) pp. 134–40.
17. *Chiu chang suan shu*, chapter 8, problem 13:

    There is a common well belonging to five families; (if we take) 2 lengths of rope of family X, the remaining part equals 1 length of rope of family Y; the remaining part from 3 ropes of Y equals 1 rope of Z; the remaining part from 4 ropes of Z equals 1 rope of V; the lacking part remaining from 5 ropes of V equals 1 rope of U; the remaining part from 6 ropes of U equals 1 rope of X. In all instances if one gets the missing length of rope, the combined lengths will reach (the water). Find the depth of the well and the length of the ropes.

    If we let W equal the depth of the well, the following system of equations result:

$$\begin{aligned} 2X+Y&=W\\ 3Y+Z&=W\\ 4Z+V&=W\\ 5V+U&=W\\ 6U+X&=W \end{aligned}$$

    which are readily reduced to:

$$\begin{aligned} 2X-2Y-Z&=0\\ 2X+Y-4Z-V&=0\\ 2X+Y-5V-U&=0\\ X+Y-6U&=0 \end{aligned}$$

18. *Sun Tzu suan ching*, chapter 3, problem 10.
19. See the discussion of the Chinese Remainder Theorem in Oystein Ore, *Number Theory and its History*, (New York: McGraw-Hill Inc., 1948) pp. 245–49.