The Utility of Recreational Mathematics*

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Prelude and Vademecum

This article is an amplification with illustrations of a write-up of my talk at the University Mathematics Teaching Conference at Sheffield Hallam University, 7 September 1999 [Singmaster 2000].

That talk was based on earlier talks that I had given on this topic, notably at the European Congress of Mathematicians in 1992 [Singmaster 1992; 1996]. Some topics discussed there were skipped in my talk at Sheffield, and some brief remarks on a few topics not mentioned at Sheffield were added in the write-up of that talk [Singmaster 2000]. I have now combined the material from all of these talks into the following, accompanied by suitable images.

> Les hommes ne sont jamais plus ingénieux que dans l'invention des jeux.
>
> [Men are never more ingenious than in inventing games.]
>
> —Leibniz to De Montmort, 29 Jul 1715.

> Amusement is one of the fields of applied mathematics.
>
> —William F. White, *A Scrap-Book of Elementary Mathematics*, 1908.

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> ... [I]t is necessary to begin the Instruction of Youth with the Languages and Mathematicks. These should ... be taught to-gether, the Languages and Classicks as ... Business and the Mathematicks as ... Diversion.
> —Samuel Johnson, first President of Columbia University, in 1731.

Introduction

My title is a variation on Eugene Wigner's famous essay "The unreasonable effectiveness of mathematics in the physical sciences" [1960]. Like Wigner, I originally had "unreasonable" in my title and did not come up with any explanation, but I believe that I have an explanation that makes it reasonable. But first let me describe the background and illustrate the situation. For about 25 years, I have been working to find sources of classical problems in recreational mathematics. This has led to an annotated bibliography/history of the subject [Singmaster 2013], now covering about 470 topics on almost 1,000 pp, where you can find more details about the topics discussed in this article.

The Nature of Recreational Mathematics

To begin with, it is worth considering what is meant by recreational mathematics. An obvious definition is that it is mathematics that is fun. However, almost any mathematician enjoys the work, even in studying eigenvalues of elliptic differential operators; so this definition would encompass almost all mathematics and hence is too general. There are two, somewhat overlapping, definitions that cover most of what is meant by recreational mathematics.

- Recreational mathematics is mathematics that is fun and popular—that is, the problems should be understandable to the interested lay person, though the solutions may be harder. (However, if the solution is too hard, this may shift the topic from recreational toward the serious—e.g. Fermat's Last Theorem, the Four Colour Theorem or the Mandelbrot Set.)

- Recreational mathematics is mathematics that is fun and used pedagogically either as a diversion from serious mathematics or as a way of making serious mathematics understandable or palatable. These pedagogic uses of recreational mathematics are already present in the oldest known mathematics and continue to the present day.

In both cases, the fun aspect is often accentuated by posing the problem in a context that is illegal, immoral, or politically incorrect (for one or more reasons), as well as being highly unlikely or even downright impossible. This whimsey is actually important, in that it makes the problem memorable; and the artificiality often eliminates unnecessary complications that tend to occur in reality. Further, the problem may be illustrated or even encapsulated in a physical object that one can see and touch—I am particularly fond of such problems and will cite several examples below.

Mathematical recreations are as old as mathematics itself, and we will later see some prehistoric examples. The earliest piece of Egyptian mathematics, the Rhind Papyrus of ca. −1800, has a problem (No. 79) where there are 7 houses, each house has 7 cats, each cat ate 7 mice, each mouse would have eaten 7 ears of spelt (a kind of wheat), and each ear of spelt would produce 7 *hekat* (a unit of volume) of spelt. Then $7 + 49 + 343 + 2401 + 16807$ is computed. A similar problem of adding powers of 7 occurs in Fibonacci (1202) [Sigler 2002], in a few later medieval texts, and in the children's riddle rhyme "As I was going to St. Ives." Despite the gaps in the history, it is tempting to believe that "St. Ives" is a descendant from the ancient Egyptians. Though there is some question as to whether this problem is really a fanciful exercise in summing a geometric progression, it has no connection with other problems in the papyrus and seems to be inserted as a diversion or recreation. (See Figures 1–3.)

Figure 1: Rhind papyrus No. 79.

The earliest mathematical works from Babylonia also date from about −1800 and they include such problems as the following on AO 8862 (see Figure 4).

> "I know the length plus the width of a rectangle is 27, while the area plus the difference of the length and the width is 183. Find the length and width."

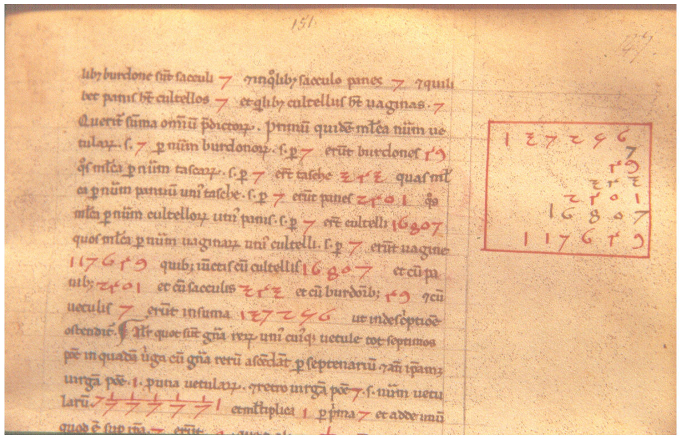


Figure 2: Extract from Fibonacci's *Liber abbaci*.

By no stretch of the imagination can this be considered a practical problem! Rather it is a way of presenting two equations in two unknowns, leading to a quadratic equation, in an effort to make solving the latter more interesting for the student.

These two aspects of recreational mathematics—the *popular* and the *pedagogic*—overlap considerably, and there is no clear boundary between them and "serious" mathematics. In addition, there are several other independent fields that contain much recreational mathematics: games; mechanical puzzles; magic; art.

Games of chance and games of strategy also seem to be about as old as human civilization. The mathematics of games of chance began in the Middle Ages, and its development by Fermat and Pascal in the 1650s rapidly led to probability theory. Insurance companies based on this theory were founded in the mid-18th century. The mathematics of games of strategy started only about the beginning of the 20th century, but soon developed into game theory, both of the von Neumann-Morgenstern type and later of the Conway type.

Mechanical puzzles range widely in mathematical content. Some only require a certain amount of dexterity and three-dimensional ability; others require ingenuity and logical thought; while others require systematic application of mathematical ideas or patterns, such as Rubik's Cube, the Chinese Rings, the Tower of Hanoi, and Rubik's Clock.

Much magic has a mathematical basis that the magician uses but carefully conceals—e.g., the fact that the opposite faces of a die add up to 7; binary divination; the fact that the period of a perfect (faro or riffle) shuffle of a 52-card pack of cards is 8.

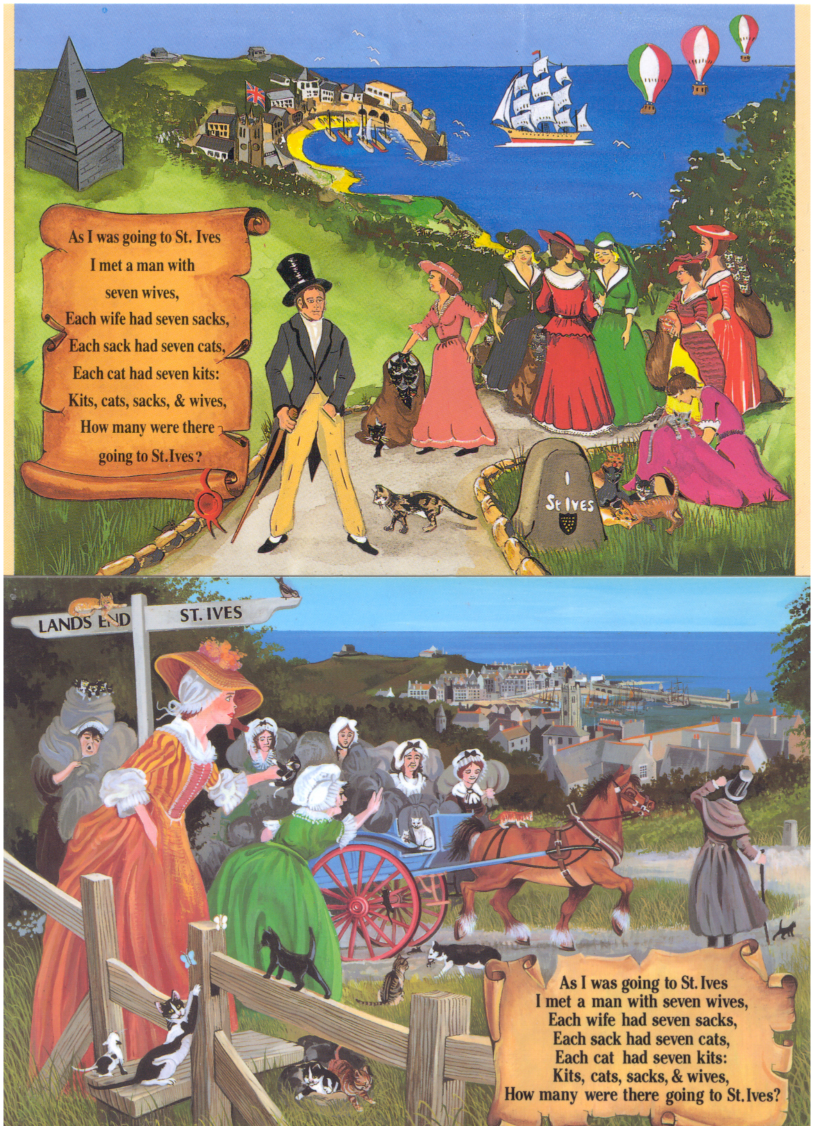


Figure 3: Postcards illustrating the St. Ives children's riddle rhyme.

The creation of beauty often leads to questions of symmetry and geometry that are studied for their own sake—e.g., the carved stone balls that we will see later.

This outlines the conventional scope of recreational mathematics, but there is some variation due to personal taste.

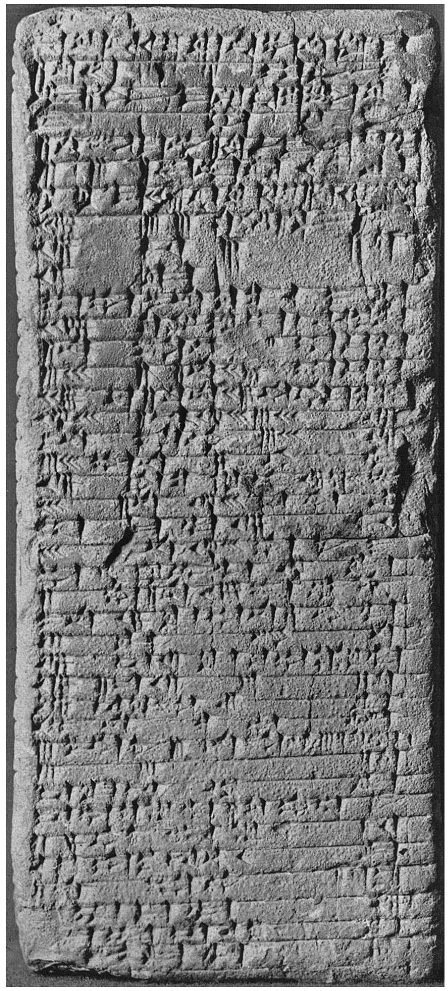


Figure 4: Babylonian tablet AO 8862, Face I, in the Louvre. Reproduced full size in print version of this issue of *The UMAP Journal*. Source: Thureau-Dangin [1932, Plate I].

The Utility of Recreational Mathematics

How is recreational mathematics useful?

- **Recreational problems are often the basis of serious mathematics.** The most obvious fields are probability and graph theory, where popular problems have been a major (or even dominant) stimulus to the creation and evolution of the subject. Further reflection shows that number theory, topology, geometry, and algebra have all been strongly stimulated by recreational problems. (Though geometry has its origins in practical surveying, the Greeks treated it as an intellectual game; and much of their work must be considered as recreational in nature, even though they viewed it more seriously, as reflecting the nature of the world. From the time of the Babylonians, algebraists tried to solve cubic equations, though they had no practical problems that led to cubics.) There are even recreational aspects of calculus—e.g., the many curves studied since the 16th century. Consequently, the study of recreational topics is necessary to understanding the history of many, perhaps most, topics in mathematics.

Before Aristotle, the Greeks used logic as a game of forcing an opponent to accept your conclusions, but had never formalized the rules. Aristotle began the study of logic in order to formalize the rules of this game.

- **Recreational mathematics has frequently turned up ideas of genuine but non-obvious utility.** I will mention a few examples later.

  Such unusual developments, and the more straightforward developments of the previous point, demonstrate the historical principle of "The (unreasonable) utility of recreational mathematics." This and similar ideas are the historical and social justification of mathematical research in general and for the study of recreational mathematics, and I will return to this point later.

- **Recreational mathematics has great pedagogic utility**, and this will be the main theme of my examples.

- **Recreational mathematics is very useful to the historian of mathematics.** Recreational problems often are of great age and usually can be clearly recognised; they serve as useful historical markers, tracing the development and transmission of mathematics (and culture in general) in place and time. The Chinese Remainder Theorem, magic squares, the Cistern Problem, and the Hundred Fowls Problem are excellent examples of this process.

  > The original Hundred Fowls Problem, from 5th century China, has a man buying 100 fowls for 100 *cash* (an old coin). Roosters cost 5, hens 3, and chicks are 3 for a *cash*—how many of each did he buy?

  The number of topics that have their origins in China or India is surprising and emphasises our increasing realisation that modern algebra and arithmetic derive more from Babylonia, China, India, and the Arabs than from Greece.

Some Examples of Useful Recreational Mathematics

I outline examples to show how recreational mathematics has been useful. (I stretch "recreational" a bit to include some other non-practical topics.)

From Gambling Bets to the Insurance Industry

The theory of probability and statistics grew from the analysis of gambling bets to the basis of the insurance industry in the 17th and 18th centuries.

Much of combinatorics likewise has its roots in gambling problems. The theory of Latin squares began as a recreation but has become an important technique in experimental design (and then returned again in connection with Sudoku puzzles).

From Euclid to the Moon and to Buckyballs

Greek geometry, though it had some basis in surveying, was largely an intellectual exercise, pursued for its own sake. The conic sections were developed with no purpose in mind, but 2000 years later turned out to be just what Kepler and Newton needed and were what took men to the Moon.

The regular, quasi-regular, and Archimedean polyhedra were developed long before they became the basis of molecular structures. Indeed, the regular solids are now known to be prehistoric. Beginning in 1985, chemists became excited about fullerenes, molecules of carbon in various polyhedral shapes, of which the archetype is the truncated icosahedron, with 60 carbon atoms at the vertices, named *buckminsterfullerene* after Buckminster Fuller (1895–1983), a proponent of geodesic domes. Spherical fullerenes are consequently nicknamed "buckyballs" (see Figure 5). Such molecules apparently are the basis for the formation of soot particles in the air. The idea of making such molecules seems to have originated with David Jones, the scientific humorist who writes as "Daedalus," in one of his humour columns. Chemists have also synthesized hydrocarbons in the shapes of a cube (cubane, C_8H_8, in 1964) and a dodecahedron (dodecahedrane, $C_{20}H_{20}$, in 1982).

From Non-Euclidean Geometry to Geometry of Physical Space

Non-euclidean geometry was developed long before Einstein considered it as a possible geometry for space.

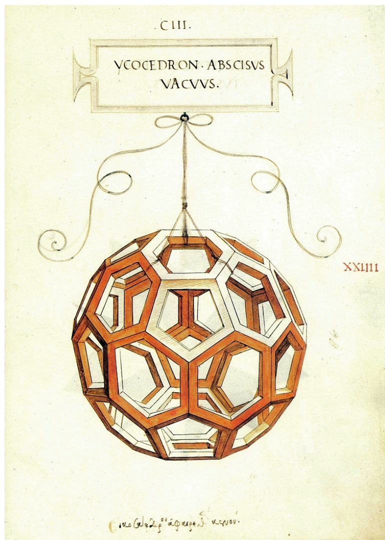


Figure 5: Frame of a truncated icosahedron buckyball, C_{60}, illustrated in the Ambrosiana manuscript of Luca Pacioli's *De divina proportione* (1509) by Leonardo da Vinci. Plate XXIIII, folio 103 recto. Source: Veneranda Biblioteca Ambrosiana, DeAgostini Picture Library, Scala, Florence.

From River-Crossing Puzzles to Graph Theory

The river-crossing problems and the problem of getting camels across a desert, which occur in Alcuin, ca. 800, are considered to be the earliest combinatorial optimization problems. Such problems are now solved by graph-theoretic methods, dynamic programming, or integer programming.

The problem of the Seven Bridges of Königsberg (Figure 6), mazes, knight's tours, and circuits on the dodecahedron (the Icosian Game) (Figures 7 and 8) were major sources of graph theory and are the basis of major fields of optimization, leading to one of the major unsolved problems of the century: Does P = NP? The routes of postmen, streetsweepers and snowplows, as well as those of salesmen, are worked out by these methods. Further, Hamilton's thoughts on the Icosian Game led him to the first presentation of a group by generators and relations (Figure 9).

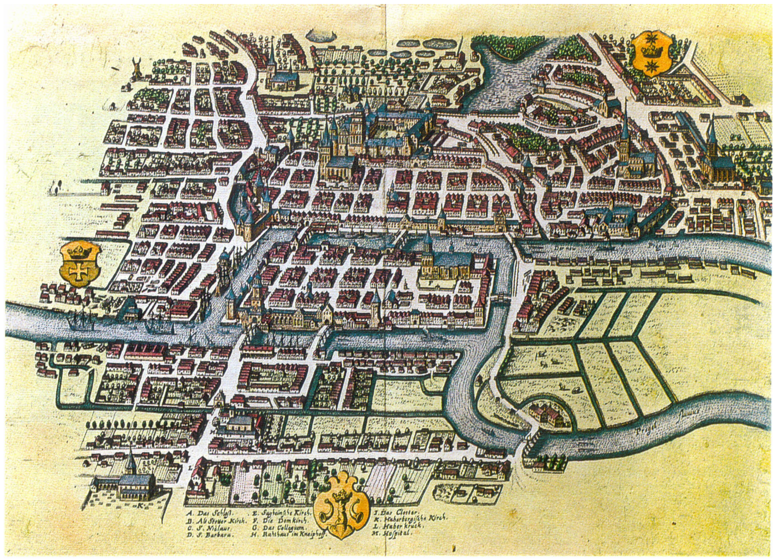


Figure 6: Coloured map of the city of Königsberg ca. 1641, showing the seven bridges featured in the problem solved by Euler. Copper engraving by Matthäus Merian the Elder or Merian Erben (his sons); colourizer unknown. Source for uncoloured engraving: Zeiller [1652].

From Number Theory to Splicing Phone Cables

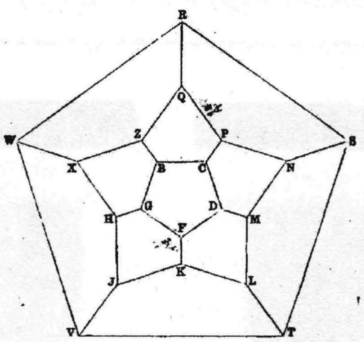
Number theory is another of the fields where recreations have been a major source of problems, and these problems have been a major source for modern algebra. Fermat's Last Theorem led to Kummer's invention of ideals and most of algebraic number theory. There was a famous application of primitive roots to the splicing of telephone cables to minimize interference and crosstalk [Rosen 1984, 280–286; 2005, 397–399]. Primality and factorization were traditionally innocuous recreational pastimes; but since 1978 when Rivest, Shamir, and Adleman introduced their method of public-key cryptography (now known as RSA cryptography), my friends in this field get rung up by reporters wanting to know if the national security is threatened because someone has factored a large number. The factorization of a big number or the determination of the next Mersenne prime are generally front-page news now.

From Buying a Horse to Negative Numbers

A major impetus for algebra has been the solving of equations. The Babylonians already gave quadratic problems where the area of a rectangle was added to the difference between the length and the width. This clearly

THE ICOSIAN GAME.

Entered at Stationers' Hall.



Registered agreeably to Act V. & VI. Vic. cap. 100.

IN this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., &c., of Dublin, and by him named *Icosian*, from a Greek word signifying "twenty") a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed *along the lines* of the figure, and also to fulfil certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in the hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

LONDON:

PUBLISHED AND SOLD WHOLESALE BY JOHN JAQUES AND SON, 102 HATTON GARDEN;

AND TO BE HAD AT MOST OF THE LEADING FANCY REPOSITORIES THROUGHOUT THE KINGDOM.

Figure 7: Advertisement with instructions for Hamilton's Icosian Game. Only four original boards of the plane version are known to exist.

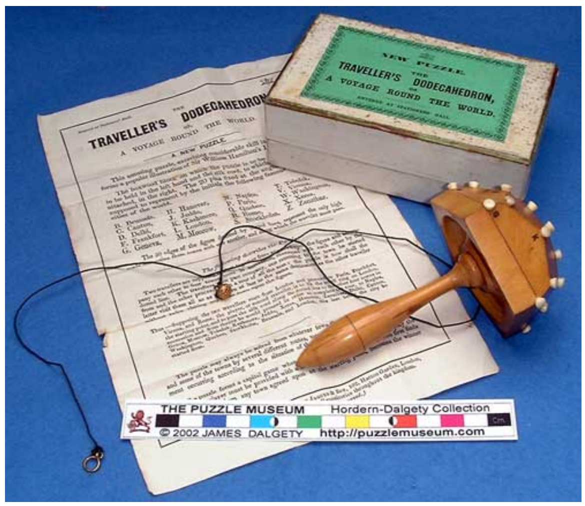


Figure 8: The only known remaining instance of the Traveller's Dodecahedron, a revision by Hamilton of his Icosian game with simpler rules. The 30 edges on the head represent roads to use to visit 20 ivory pegs that represent cities. "Two travellers set off visiting 4 neighbouring towns. One returns home and the other continues to travel around the world trying to visit all the remaining cities once only." Photo courtesy of James Dalgety. ©Copyright 2013 Hordern-Dalgety Collection. `http://puzzlemuseum.com`, `http://puzzlemuseum.com/month/picm02/200207icosian.htm`.

Sir William Rowan Hamilton read a Paper on a new System of Roots of Unity, and of operations therewith connected: to which system of symbols and operations, in consequence of the geometrical character of some of their leading interpretations, he is disposed to give the name of the "ICOSIAN CALCULUS."

This Calculus *agrees* with that of the Quaternions, in three important respects: namely, 1st, that its three chief symbols, ι, κ, λ, are (as above suggested) *roots* of *unity*, as *i*, *j*, *k* are certain *fourth roots* thereof: 2nd, that these new roots *obey* the *associative* law of multiplication; and 3rd, that they are *not* subject to the *commutative* law, or that their *places* as *factors* must *not* in general be *altered* in a *product*. And it *differs* from the Quaternion Calculus, 1st, by involving roots with *different exponents;* and 2ndly, by *not requiring* (so far as yet appears) the *distributive* property of multiplication. In fact, + and −, in these new calculations, enter *only* as *connecting exponents*, and *not* as connecting *terms:* indeed, *no terms*, or in other words, *no polynomes*, nor even binomes, have hitherto presented themselves, in these late researches of the author. As regards the *exponents* of the new roots, it may be mentioned that in the *principal system*,— for the new Calculus involves a *family of systems*,—there are adopted the equations,—

$$1 = \iota^2 = \kappa^3 = \lambda^5, \ \lambda = \iota\kappa; \qquad \text{(A)}$$

so that we deal, in it, with a *new square root*, *cube root*, and *fifth root*, of *positive unity;* the latter root being the *product* of the two former, when taken in an *order* assigned, but *not* in the opposite order. From these simple assumptions (A), a long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timæus; namely, the Icosaedron, and the Dodecaedron: whereof the *angles* may *now* be *unequal*. By making $\lambda^4 = 1$, the author obtains other symbolical results, which are interpreted by the Octaedron and the Hexaedron. The Pyramid is, in *this* theory, almost too simple to be interesting: but it is dealt with by the assumption, $\lambda^3 = 1$, the other equations (A) being untouched. As one fundamental result of those equations (A), which may serve as a slight specimen of the rest, it is found that if we make $\iota\kappa^2 = \mu$, we shall have

$$\mu^5 = 1, \ \mu = \lambda\iota\lambda, \ \lambda = \mu\iota\mu;$$

so that this *new fifth root* μ has relations of perfect *reciprocity* with the former fifth root λ. But there exist more *general* results, *including* this, and others, on which Sir W. R. H. hopes to be allowed to make a future communication to the Academy: as also on some applications of the principles already stated, or alluded to, which appear to be in some degree interesting.

Figure 9: Hamilton's mathematics inspired by the Icosian Game, as presented by him in *Proceedings of the Royal Irish Academy* 6 (1858) 415–416. Communicated November 10, 1856. Hamilton's collected papers notes that "[t]his is the substance of a letter written on 27 October 1856 to the Rev. Charles Graves, D.D." (*The Mathematical Papers of Sir William Rowan Hamilton, Volume III: Algebra*, edited by H. Halberstam and R.E. Ingram, 609. New York: Cambridge University Press, 1967.)

had no practical significance. Similar impractical problems led to cubic equations and the eventual solution of the cubic. Negative solutions first become common in medieval puzzle problems about men buying a horse or finding a purse.

Galois fields and even polynomials over them are now standard tools for cryptographers.

Recreational Curves to Analysis

Even in analysis, the study of curves (e.g., the cycloid) had some recreational motivation.

From Knots to DNA

Topology has much of its origins in recreational aspects of curves and surfaces. Knots, another field once generally considered of no possible use, are now of great interest to molecular biologists who have discovered that DNA molecules form into closed chains which may be knotted, or not knotted.

The Möbius strip arose about 1858 in work by both August Ferdinand Möbius (1790–1868) and Johann Benedict Listing (1808–1882), Listing being apparently a bit earlier. Depictions of it occur in Roman mosaics (Figure 10, noticed by Charles Seife in 2002), including a strip with five half-turns (Figure 11) [Larison 1973]. These two examples, together with other examples of "early" Möbius strips, are discussed in Cartwright and González [2016].

Figure 10: Möbius strip in detail from a floor mosaic from a Roman villa near Sentinum (now Sassoferrato, Umbria), ca. 200 A.D., now in the Glyptothek in Munich, Germany. Depicted is Aion, god of Eternity, surrounded by a zodiac wheel, with earth mother Tellus seated. Source: Photograph in the public domain by Bibi Saint-Pol.

By 1890, the Möbius strip was being used as a magic trick, magic being another application of mathematics; indeed, some people view all mathematics as magic! More recently, such strips have served as the basis for works by M.C. Escher, art being yet another application of mathematics.

The Möbius strip has also been patented several times! e.g., as a single-sided conveyor belt that has double the wearing surface (Figure 12).

There are a number of other practical uses for the Möbius strip, but the most unusual is as a non-inductive electrical resistor (Figure 13).

None of the patents that I have seen make any reference to any previous occurrence of the concept. Martin Gardner says it has also been patented as a non-inductive resistor. Those who still have dot-matrix printers may (or may not) know that printer ribbons commonly have a twist (so they

Figure 11: Möbius strip with five half-turns in a Roman floor mosaic of Orpheus charming the animals, ca. 200 A.D., now in the Museum of Pagan Art, Arles, France. Source: Detail from Finoskov, Creative Commons (CC BY SA-3.0), Wikimedia Commons.

are Möbius strips!) in order to allow the printer to use both edges. I first discovered this when I found one of our technicians trying to put such a ribbon back into its cartridge; he had done it several times, but it kept coming out twisted, which he thought was his mistake!

From Chinese Rings to Binary Codes

Gray Codes

The Chinese Rings puzzle (Figure 14), known as *bagenaudier* ("time-waster") in French, may indeed have originated in China 1,800 years ago.

In combinatorics, the pattern of solution of the Chinese Rings puzzle is the binary coding known as the *Gray code*, patented as an error-minimising code by Frank Gray (1887–1969) of Bell Labs in 1953 (Figure 15) and already used in the same way by Émile Baudot (1845–1903) in the 1870s [Baudot 1879] in inventing the predecessor of the teletype (it is from Baudot's code that the term "baud" arose as a measure of transmission speed).

Chain Codes

Another binary coding, sometimes called a *chain code*, was used by Sanskrit poets in about 1000 to memorise all the combinations of long and short syllables [Stein 1961; 1976]. The 10 syllables in the Sanskrit nonsense word

ya-mā-tā-rā-ja-bhā-na-sa-la-gām

contain in successive groups of three all the triplets of long beats (marked with a bar over the a) and short beats (unmarked a). Moreover, since the last two syllables are the same as the first two, if we regard the sequence

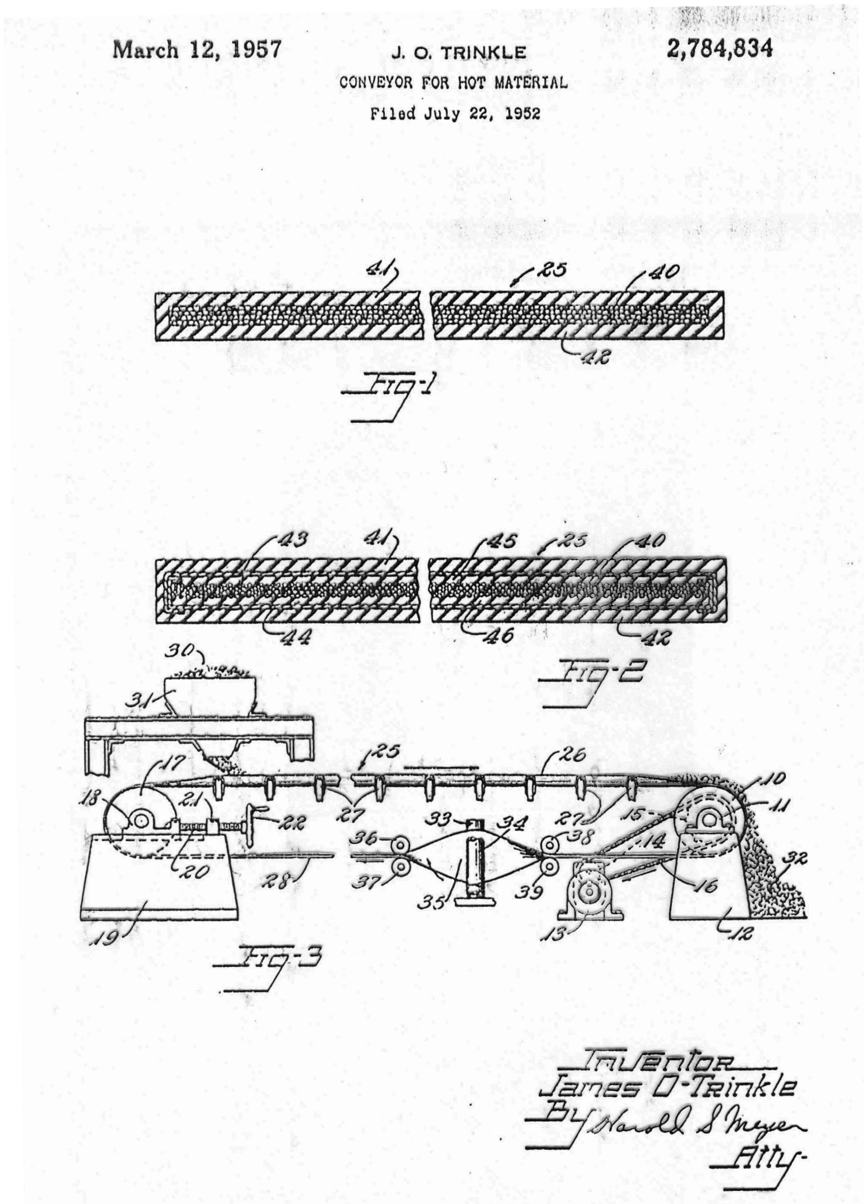


Figure 12: Image from a patent for a single-sided conveyor belt.

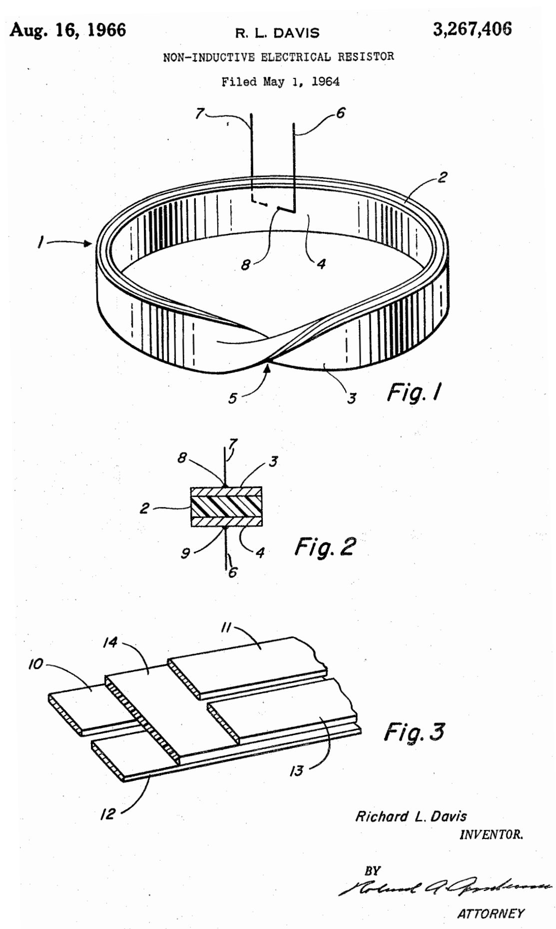


Figure 13: Patent for use of a Möbius strip as a non-inductive electrical resistor, by Richard L. Davis, granted in 1966. U.S. Patent 32674906 A.

Figure 14: Chinese Rings puzzles. The task is to remove all the rings.

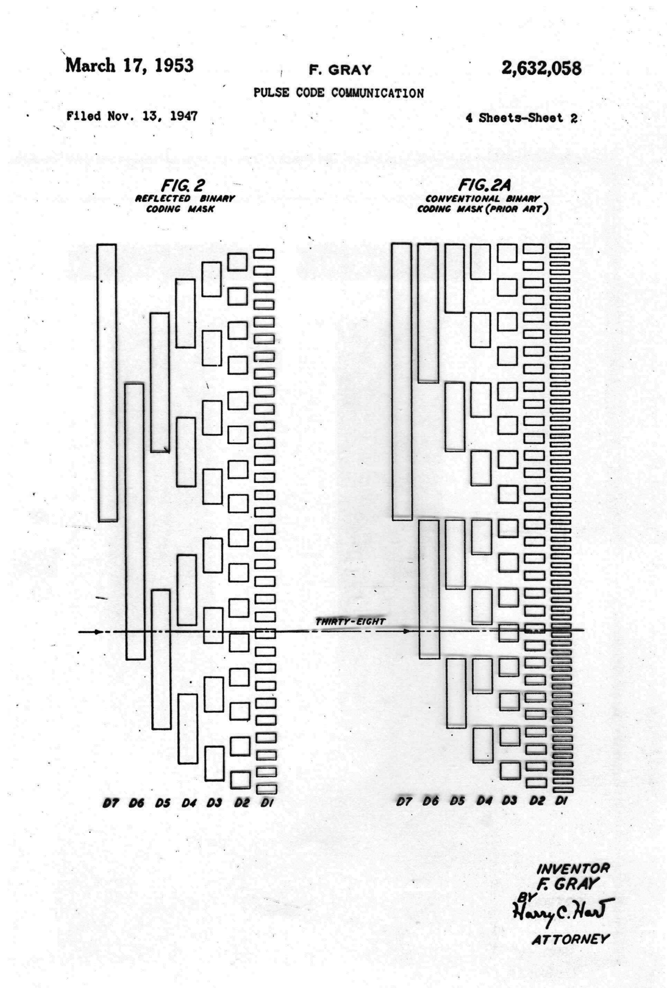


Figure 15: Figures from Gray's patent for pulse code communication.

of syllables as wrapping around, we could arrange the syllables in the form of a wheel.

Baudot redesigned his printing telegraph to use a chain code [Heath 1961, 540; 1972, 83; Baudot 1895]. The idea of a chain code led to the more general mathematical concept of a *de Bruijn sequence* [Gurudev 2007; Diaconis and Graham 2012, 42–60]. In a de Bruijn sequence, every possible subsequence of a prescribed length from an alphabet of characters appears exactly once in the sequence, which like a memory wheel cycles back on itself. For example, the de Bruijn sequence

$$0\ 0\ 0\ 1\ 0\ 1\ 1\ 1$$

contains in order all the different subsequences of length 3:

000, 001, 010, 101, 011, 111, 110, 100.

Such codes are painted on factory and warehouse floors to enable robots to determine where they are by scanning a small section of the floor. They have also been used as the basis of card tricks—divinations—where the values of cards are determined from a small amount of information [Diaconis and Graham, 25–29, 42–60]. Diaconis and Graham note some confusion of Gray codes and chain codes:

> [Magicians] mistakenly call de Bruijn sequences "Gray codes." ... But as far as we know, there has never been a single use [of Gray codes] in magic. [Diaconis and Graham 2012, 25]

The earliest mention of what later became known as chain codes and de Bruijn sequences seems to be by Flye Saint-Marie [1894], though Baudot's use of it for a teleprinter dates from about 1882, with equipment using it exhibited in 1889 [Heath 1961, 540; 1972, 83]. Kerr [1961] offers in brief some mechanical details of a production teletype machine that used a chain code.

Examples of Recreational Mathematics with Objects

Several of these examples are based on objects that I passed around at the lectures.

Neolithic Polyhedra

These "carved stone balls" date from ca. −2500, and occur in eastern Scotland. Examples are in the Royal Scottish, Ashmolean, Dundee, and Aberdeen Museums. Figure 16 shows a resin model of a carved stome ball from Aberdeenshire, made by an artist in Glastonbury. No one knows the purpose of these.

Plimpton 322, ca. −1800

This is the famous Old Babylonian tablet listing Pythagorean triples. Some years ago I persuaded Columbia University to make casts from the original, and Figure 17 is a photograph of one of those.

Figure 16: A resin model of a neolithic carved stone ball (about 9 cm across).

Figure 17: Facsimile of the Old Babylonian tablet Plimpton 322 that lists Pythagorean triples.

Roman Dodecahedron, ca. 200–400

Approximately 100 of these are known, from Roman sites north of the Alps. The one shown in Figure 18 was found in 1939 in Tongeren, Belgium, and dates to 150-400 A.D. [Huylenbrouck 2012]. Its total height is 81 mm; the height without the balls at the corners is 66 mm. I have seen a somewhat smaller example at the Hunt Museum in Limerick. No one knows their purpose [Guggenheim 2013]. Nevertheless, they admirably match the description of Hamilton's Traveller's Dodecahedron of Figure 8 on p. 14.

Figure 18: Roman dodecahedron found at Tongeren, Belgium, in 1939 and now situated in the Gallo-Roman Museum Tongeren. Reproduced full size in print version of this issue of *The UMAP Journal*. Photo by Guido Schalenbourg, ©Gallo-Roman Museum Tongeren, with thanks to Else Hartoch, Collection Management Coordinator / Research.

Chinese Magic Square

The cast-iron facsimile in Figure 19 (cast at reduced size) is one of the five cast-iron examples of a 6×6 magic square excavated near Xi'an, China, in 1956. It is inscribed in East Arabic numerals (similar to those still used in the Middle East) and dates to the Yuan Dynasty (1280–1368) [Li and Du 1987, 172].

Examples of Medieval Problems

Fibonacci Numbers

Figure 20 shows a page from the manuscript ca. 1275 at Siena of Fibonacci's *Liber abbaci* of 1202 and 1228. This manuscript, which also includes his hand signs for numbers (Figure 21), is apparently the earliest known extant version of his book. The page shows the Fibonacci sequence 1, 2, 3, 5, ...

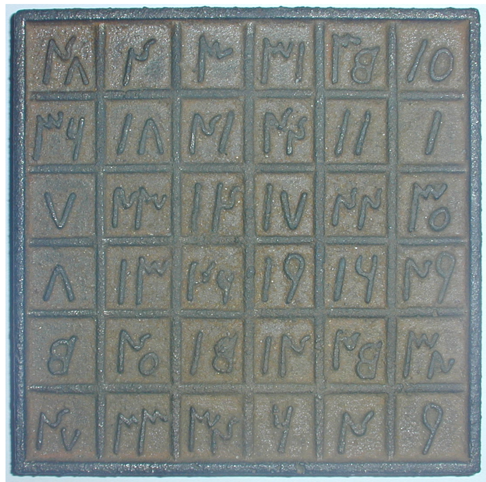


Figure 19: Cast-iron facsimile of a Chinese magic square dating to the Yuan Dynasty (1280–1368).

377, where each entry is the sum of the two preceding. Fibonacci introduced the sequence in connection with a fanciful model for the number of rabbit pairs in successive generations.

The Fibonacci numbers were known to ancient Sanskrit poets, from an uncertain date about 2000 years ago. The number of different patterns of fixed length of long syllables and short syllables, where a long syllable is twice as long as a short syllable, is a Fibonacci number. For example, the patterns with total length the equivalent of 4 short syllables are LL, SSL, SLS, LSS, and SSSS, for a total of 5. However, the first Indian work in which mathematical investigation was made of such numbers was not until 1356, where they were related to binomial coefficients [Singh 1985; Knuth 2011, 47ff].

The Josephus Problem

This is the problem of recursively counting out every k-th person from a circle of n people. Early versions counted out half the group (see Figure 22); later authors and the Japanese are interested in the last man—the survivor. Euler (1775) seems to be the first to ask for the last man in general. Cardan (1539) is the first to associate this process with Josephus; some later authors derive this from the Roman practice of decimation.

> According to Josephus's account of the siege of Yodfat (in the First Jewish War against the Romans, in 66–73), he and his 40 soldiers were trapped in a cave, the exit of which was blocked

Figure 20: Earliest known presentation of the Fibonacci sequence of numbers, in Fibonacci's *Liber abbaci* (ca. 1275).

Figure 21: Fibonacci's version of Roman hand signs for numerals.

Figure 22: Calandri's version of the Josephus problem, with 15 each of Franciscans (in brown) and Camoldensians (in white) on a boat, and counted out by $k = 9$. Where should the standing monk start counting by 9, and in which direction, so that all the white-robed monks are counted out?

> by Romans. They chose suicide over capture and decided that they would form a circle and start killing themselves using a step of three. Josephus states that by luck or possibly by the hand of God, he and another man remained the last and gave up to the Romans. —Wikipedia [2014b]

However, Josephus's own account does not mention a step of three, only drawing of lots [Josephus ca. 75].

Right-Triangle Problems

Right-triangle problems date back to Old Babylonian (−1800), Chinese (ca. −150?), and Indian sources. The Indians include Bhaskara I (629), Mahavira (850), Chaturveda's 860 commentary on Brahmagupta, and Bhaskara II (1150).

The Sliding Spear

The Sliding Spear (= Leaning Reed) Problem goes back to Old Babylonian times (Figure 23).

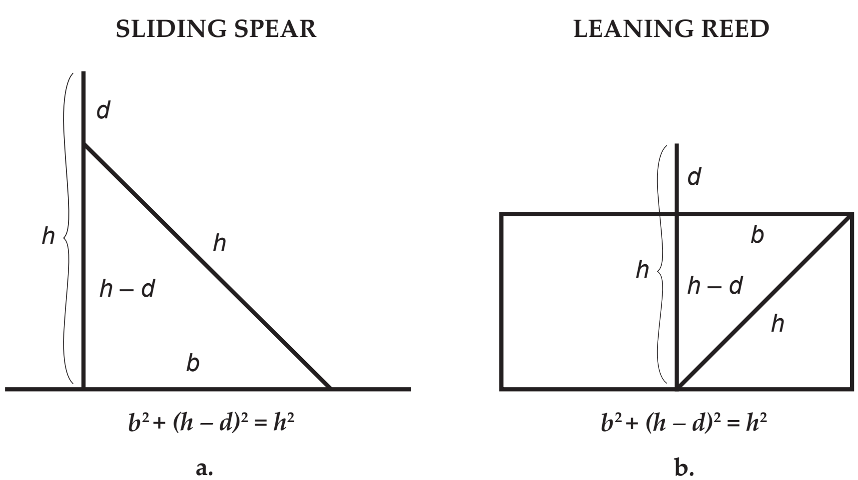


Figure 23: Diagrams of the Sliding Spear Problem and Leaning Reed Problem.

The Broken Tree Problem

The Broken Tree (or Bamboo) (= Hawk and Rat = Peacock and Serpent) Problem goes back to a Chinese source:

> A bamboo (or tree) of height H breaks at height X from the ground so that the broken part reaches from the break to the ground at distance D from the foot of the bamboo; H and D are given and X is sought.

The Two Towers Problem

The Two Towers Problem goes back to Bhaskara I (629), who attributes it to earlier writers! In ca. 1370, dell'Abbaco introduces the following variation:

> Given two towers of heights H_1 and H_2, situated a distance D apart with a rope of length L between the tower tops. How high H from the ground does a sliding weight on the rope hang—or does it reach to the ground?? (See Figure 24.)

Dell'Abbaco gives: $H_1 = 60$, $H_2 = 40$, $D = 40$, $L = 110$, and claims $H = 0$.

A general solution can be tedious; but if one finds the solution by brute force, the form of the solution shows that it is easy to find!

River-Crossing Problems

The *Propositiones ad acuendos juvenes*, attributed to Alcuin of York, ca. 800, contains two classic river-crossing problems: wolf, goat, and cabbage; and the three couples.

Wolf, Goat, Cabbage Problem

In the first, a farmer must transport a wolf, a goat, and a cabbage across the river in a boat that can hold only the farmer and one other item; the restrictions are that the wolf cannot be left on either bank with the goat, nor the goat with the cabbage, unaccompanied by the farmer. (See Figure 25.).

Three Couples Problem

In the three couples (or "jealous husbands") problem, three married couples must cross the river in a boat that can hold only at most two people; the constraint is that no wife can be in the boat or on either bank unless her husband is present.

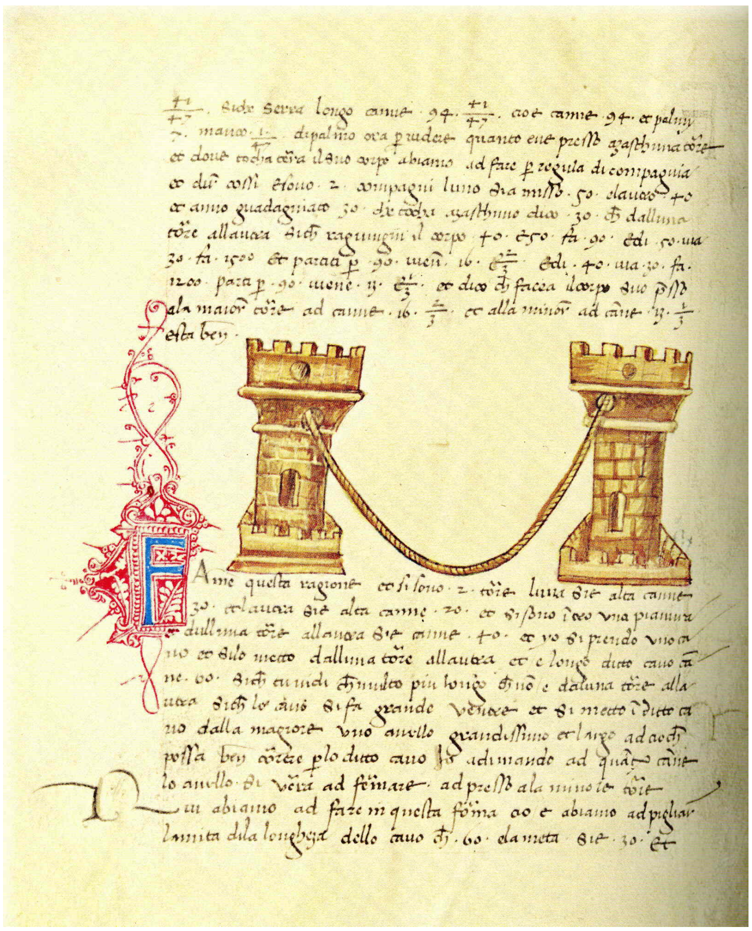


Figure 24: The Two Towers Problem in Pietro Paolo Muscarello's *Algorismus* MS of 1478.

Figure 25: Wolf-goat-grain puzzle from *Columbia Algorism*, anonymous Italian MS, ca. 1350 [Vogel 1977, 130–131, 191; Cowley 1923, 402 and plate opposite].

Modern Combinatorial Optimization

The problems above are among the earliest combinatorial optimization problems. Martin Grötschel in Berlin uses the wolf-goat-cabbage problem to teach integer programming; his class found a shorter solution, but it involved halving the cabbage and halving the wolf! [Borndörfer et al. 1998].

To generalize the second problem requires an island in the river and remains perhaps unsolved in general, since the improved solution in Pressman and Singmaster [1989] can be criticised if one takes a more stringent jealousy condition than we did.

Examples of Modern Recreational Problems

Longest Fishpole One Can Post (Mail)

An ancient problem involves a fisherman (or hunter or skier) who wants to post (mail) his 2.5 m fishing rod (or gun or skis) and finds that the post office has a maximum parcel length of 1.5 m. The fisherman solves the problem by making a cubical box of edge 1.5 m and putting the rod in diagonally. The diagonal of the box is $1.5 \times \sqrt{3} \approx 2.598$. This is very ingenious, but unfortunately there are other postal regulations. The length plus the girth must be at most 3 m. The girth is the circumference in a plane perpendicular to the longest dimension, which is the length. For a box of dimensions $A \times B \times C$, with $A \geqslant B \geqslant C$, the girth is $2B + 2C$; and so we must have $A = 1.5$ and $A + 2B + 2C = 3$. What is the longest fishing rod that can be posted under these limitations? Suppose one uses a cylindrical mailing tube?

The problem of finding the largest *volume* that one can post (mail) is well known, and the maximum occurs for a cylindrical tube.

Crossing a Field

The following seems as if it should be an easy question, but I find it quite messy and would like to see a solution better than my own.

You are on a path which runs south to a road. Along the road is a bus stop, and you want to get to it as quickly as possible. Between the path and the road is a field; and you can cut across the field, but your speed may be slower than on the path or the road. Is it ever the case that the optimal route is to go part way along the path, then go obliquely across the field to a point part way along the road, and then go the rest of the way along the road? Try to convince yourself of the answer before doing any calculations. Determine the optimal route in general for all situations.

For standardization. let us assume that you start at the point $A = (0, W)$ and are travelling to $D = (L, 0)$ and that you travel from A to $B = (0, y)$, thence to $C = (x, 0)$, then to D (see Figure 26).

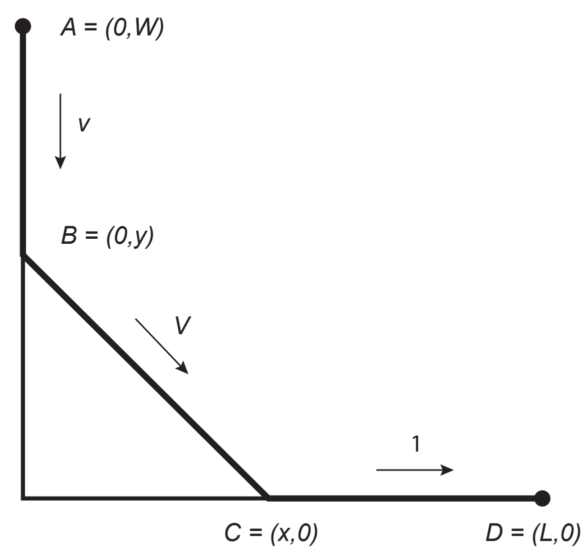


Figure 26: Diagram for the Crossing a Field Problem.

There are three speeds involved, but only their relative values are important, so you can assume that your speed on the road is a unit speed, your speed on the path is v, and your speed on the field is V, with $V \leqslant v \leqslant 1$.

There is a common feature of the problems of crossing a field and posting the longest fishpole that you should discover when you solve them, and which is why I like these problems.

Folding a Chain of Cubes

There are several versions of this puzzle on the market; I first purchased one in Paris in 1990 (see Figure 27 for one example).

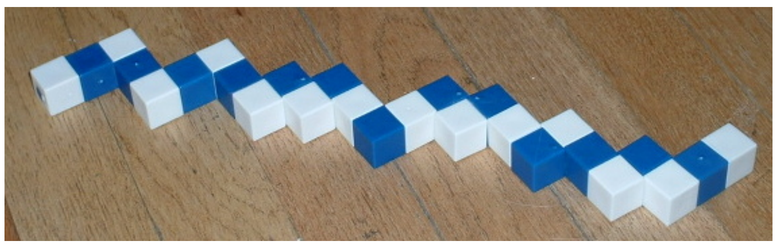


Figure 27: A snake puzzle. Photo courtesy of Eryk Vershen.

They are simple geometric analogues of "transformer" toys. Each has a chain of cubes, strung on an elastic. Each cube has a hole, either straight through or from a face to an adjacent face (effectively a right-angle bend). For a string of 27 cubes, one obviously wants to make a $3 \times 3 \times 3$ cube. How does one go about solving such problems systematically? Is there more than one solution? If so, how many solutions are there? The original version of this had one solution, but a set of five different versions has recently been marketed. The variation can be identified by specifying in order whether the connection from one cube to the next is straight (`S`) or turns (`T`)

There are also examples with the string forming a loop; for example, I have one with 36 cubes in a loop, with all pieces being bends, and making this into a $3 \times 3 \times 4$ takes a little effort. There are also examples with 64 cubes on a loop, with some straight pieces and some bent pieces—these are generally impossible to solve by hand. I also have an example with 125 pieces, all bends, which I have never solved, though it seems that a hand solution should be possible. Can you write a program to do this?

What can one say about the number of straight and bent pieces in such puzzles? Could one have a 27-cube string with all bends? Can you have versions which make several solid (or even plane) shapes?

The problem of folding the snake into a cube is equivalent to finding a hamiltonian path—that is, a path through each "cubie" (cubelet) in the cube—with the turn at each step specified in advance as either (`S`) or (`T`).

Abel et al. [2013] show that the problem of *deciding* whether it can be done at all is NP-complete, meaning that the time required to do so for an $N \times N \times N$ cube may grow more quickly than any polynomial function of N. Knowing that there is a solution, however, would not necessarily tell how to perform the transformation—but would motivate you, on recreational grounds, to try to find one!

Scherphuis [n.d.] lists all possible $3 \times 3 \times 3$ possible snake puzzles that are "doable": Not counting rotations or mirror images of the hamiltonian paths, there are 11,487 puzzles, of which 3,658 have unique solutions and 1 has 142 solutions. Scherphuis shows that it is impossible to have a doable $3 \times 3 \times 3$ puzzle with only turns (as do Ruskey and Sawada [2003]), and his Web page points to various pages with solutions for commercially available snake puzzles as well as for the Kibble Cube, a variation in which the cubes have grooves that allow for greater freedom.

A particularly clear solution to one $3 \times 3 \times 3$ puzzle is at Cole [n.d.], and there are potentially useful notations at Weston [2003] and Köller [1999]. You can find videos of people solving $3 \times 3 \times 3$ and even $4 \times 4 \times 4$ puzzles: Search the Internet with a key such as "youtube snake cube 4x4x4".

The snake puzzles have given rise to mathematical research into "bent" hamiltonian paths and cycles in any dimension, where every connection is a turn [Ruskey and Sawada 2003], and into which N^2 snakes can be folded into a flat square (whether this problem is NP-complete is unknown [Abel et al. 2013]). The general field of study is known as *combinatorial Gray codes*, in which successive objects or positions differ in some prescribed way. A recreational example is change-ringing of church bells, in which the order in which bells are rung can change only in specified ways.

But that 125-cube-long snake? I have since found the solution that came with the puzzle, and a correspondent has sent a solution.

Rubik's Cube

I spoke very briefly about Rubik's Cube, describing it as an excellent example of problem solving (Figure 28 shows the bigger version that is sometimes known as the Professor's Cube). One can identify many of the classic problem-solving skills:

- understanding the problem;
- establishing a notation;
- investigating subproblems;
- using conjugates (which is a special case of one of the basic problem-solving techniques—transform a problem to a situation one knows

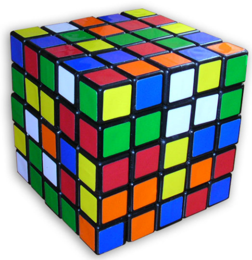


Figure 28: A larger-than-usual ($5 \times 5 \times 5$) Rubik cube, scrambled; there are even $8 \times 8 \times 8$ and $9 \times 9 \times 9$ cubes. Source: Creative Commons Attribution-Share Alike 3.0 Unported, by Maksim.

how to solve and then transform the answer back to the original situation—this is the idea of logarithms, Laplace, Fourier and other transforms, similarity transformations, change of basis, mathematical modelling, etc., as well as the idea behind machine shop or factory work (take the item to be sawn to the saw, then bring the pieces back to your workplace);

- using commutators (a less general technique, but one of great use in group theory);
- creating an algorithm;
- demonstrating completeness of the algorithm; and
- seeking an optimum algorithm (still unsolved).

The eminent Dutch puzzle designer, Oskar van Deventer, has designed and made a $17 \times 17 \times 17$ cube!

[Editor's Note: Prof. Singmaster was author of one of the first books about Rubik's Cube [1981], and is co-author of others [Slocum et al. 2009, Frey and Singmaster 2010].]

The Penrose Pieces

Penrose's pieces have led to the discovery of a new kind of solids, "quasicrystals."

I will only sketch the ideas here, with some references.

The former coat of arms (Figure 29) of London South Bank University includes "the net of half a dodecahedron," i.e., a pentagon surrounded by five other pentagons.

Figure 29: Former coat of arms of London South Bank University, designed to include two Thames barges above a pentagon surrounded by five other pentagons.

One of the basic results of crystallography is that no crystal structure can have five-fold symmetry. In 1973, I wrote to Roger Penrose on a Polytechnic letterhead that shows the half dodecahedron. Penrose had long been interested in tiling the plane with pieces that could not tile the plane periodically, and the letterhead inspired him to try to fill the plane with pentagons and other related shapes.

He soon found such a tiling with six kinds of shape and then managed to reduce it to two shapes that could tile the plane in uncountably many ways but in no periodic way. Some of the tilings have a five-fold centre of symmetry, and all have a sort of generalised five-fold symmetry. They are now called *quasicrystals*. These tilings fascinated both geometers and crystallographers and were extensively studied from the mid-1970s.

Penrose's "kites and darts" shapes were simplified further to "fat and thin rhombuses" (Figure 30). Rules for putting them together (e.g., a side corner of a kite must coincide with the tip or rear of a dart) prevent the shapes from tiling periodically. Figure 31 shows a "Penrose pattern" made from the rhombuses of Figure 30.

The rhombus shapes were extended to three dimensions, where they are related to the rhombic triacontahedron. Though the tilings are not periodic, they have quasi-axes and quasi-planes, which can cause diffraction. Using these, crystallographers determined the diffraction pattern that a hypothetical quasicrystal would produce: It has a ten-fold centre of symmetry.

In 1984, such diffraction patterns were discovered by Dan Shechtman (b. 1941) in a sample of rapidly cooled alloy now known as shechtmanite; and some 20 substances are now known to have quasi-crystalline forms. Shechtman received the 2011 Nobel Prize in Chemistry for his discovery of quasicrystals [Royal Swedish Academy of Sciences 2011a; 2011b]. In-

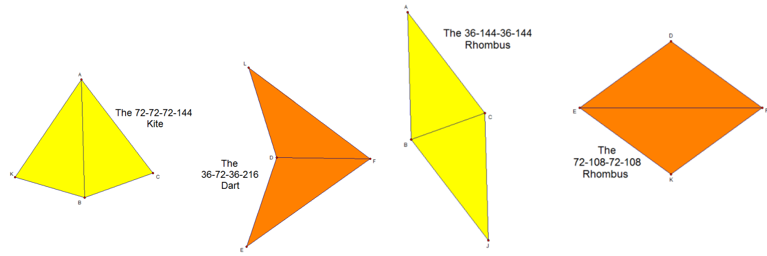


Figure 30: Penrose's dart and kite, and his fat and thin rhombuses, with notation of the degrees of the angles involved. Courtesy of Robert Austin [2014].

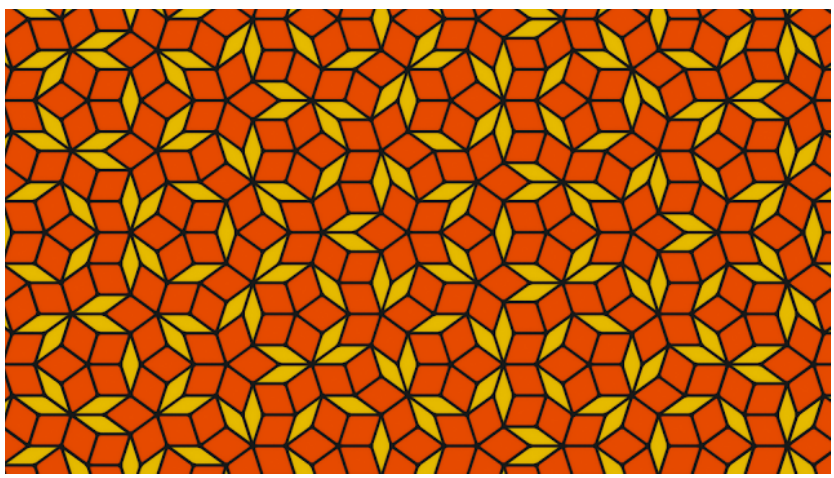


Figure 31: A Penrose pattern made from the rhombuses of Figure 30. Courtesy of Robin Wilson.

deed, examples had been found about 30 years earlier but the diffraction patterns were discarded as being erroneous! It is not yet known whether such materials will be useful but they may be harder or stronger than other forms of the alloys and hence may find use on aeroplanes, rockets, etc. So a mathematical flight of fancy has led to the discovery of a new kind of matter on which we may be flying in the future! (See Gardner [1979; 1997] for expositions of this topic.)

The Educational Value of Recreations

A Treasury of Problems

Recreational mathematics is a treasury of problems which make mathematics fun. These problems have been tested by generations going back to about 1800 BC. In medieval arithmetic texts, recreational questions are interspersed with more straightforward problems to provide breaks in the hard slog of learning. These problems are often based on reality, though with enough whimsey so that they have appealed to students and mathematicians for years. They illustrate the idea that "Mathematics is all around you ? you only have to look for it."

An Optimal Learning Experience

"A good problem is worth a thousand exercises" (ancient proverb, made up by myself). There is no greater learning experience than trying to solve a good problem. Recreational mathematics provides many such problems and almost every problem can be extended or amended. Hence recreational mathematics is also a treasury of problems for student investigations.

Solving problems naturally develops problem-solving techniques. Some of those which arise in recreational problems are:

- The problems often require clarification of the assumptions and one may vary the assumptions to get different problems.
- One may need to create a notation.
- The mathematical or logical methods needed are often non-standard and hence one has to use basic ideas in a novel way.
- The problems are often open-ended and natural generalizations are often unsolved, so one has to re-examine the problem and ask new questions.

For better or worse, mathematics is one of the only school courses where students are expected to learn how to think! But thinking, like problem solving, is best learned by doing and our problems are ideal for encouraging this.

A Communication Vehicle for History and Culture

Because of its long history, recreational mathematics is an ideal vehicle for communicating the historical and multicultural aspects of mathematics.

A Communication Vehicle for Mathematical Ideas

An additional utility of recreational mathematics is that it provides us a way to communicate mathematical ideas to the public at large.

Mathematicians tend to underestimate the public interest in mathematics. (Lee Dembart of the *Los Angeles Times* wrote that when he told people he was going to a conference on recreational mathematics, they replied that it was a contradiction in terms! And we all know the social situation when you confess that you are a mathematician and the response is, "Oh. I was never any good at maths.") Yet somewhere approaching 200 million Rubik Cubes were sold in three years! Indeed, there have been more Rubik Cubes sold in Hungary than there are people. The best-known example of a best-selling game is Monopoly, which took 50 years to sell about 90 million examples.

Another measure of the popularity of recreational mathematics is the number of books that appear in the field each year, perhaps 50 in English alone. The long-term best-selling recreational book in English must be *Mathematical Recreations and Essays* by W.W. Rouse Ball (1850–1925), originally published in 1892 and now in its 13th edition. It has rarely been out of print in that time. And there are many older books, such as *Problèmes plaisants et délectables...* by Claude Gaspard Bachet de Méziriac in 1612, which had three editions in the late 19th century, the last of which was reprinted several times in the 20th century.

Many newspapers and professional magazines run regular mathematical puzzles, though this was perhaps more common in the past. Henry Dudeney published weekly columns for about 15 years and then monthly columns for about 20 years. Martin Gardner's columns were a major factor in the popularity of *Scientific American* and probably inspired more students to study mathematics than any other influence. I have heard that circulation dropped significantly when he retired. Other major names in the field are the following:

- In English: Lewis Carroll (= Charles Lutwidge Dodgson) (1832–1898), Sam Loyd (1841–1911), "Professor Hoffmann" (= Angelo John Lewis (1839–1919) (about magic), Hubert Phillips (= "Caliban") (1891–1964), Thomas H. O'Beirne (1915–1982), Douglas St. Paul Barnard (1924–1992?), Henry Dudeney (1857–1930), Martin Gardner (1914–2010), Ian Stewart (b. 1945).

- In German: Wilhelm Ahrens (1912–1998), Hermann Schubert (1848–1911), Walther Lietzmann (1880–1959).

- In French: Édouard Lucas (1842–1891), Pierre Berloquin (b. 1939).

[I tried to carry on this tradition by contributing to the *Daily Telegraph* and to the BBC Radio 4 programme "Puzzle Panel."]

There really is considerable interest in mathematics out there; and if we enjoy our subject, it should be our duty and our pleasure to try to encourage and feed this interest. Indeed, it may be necessary for our self-preservation!

Why Is Recreational Mathematics So Useful?

I have been developing an answer to this, and it also answers Wigner's question. Mathematics has been described as a search for pattern—and that certainly describes much of what we do and also much of what most scientists do. But how do we find patterns? The real world is messy and patterns are difficult to see. As we begin to see a pattern, we tend to remove all of the inessential details and get to an ideal or model situation. These models may be so removed from reality that they become fanciful—or even recreational.

For example, physicists deal with frictionless perfectly elastic particles, weightless strings, ideal gases, etc.; mathematicians deal with random samples, exact measurements, negative money, etc. Then such models get modified and adapted into a large variety of models and techniques are developed to describe and solve them.

Now, one of the ways in which a science progresses is by seeing analogies between reality and simpler situations. For example, the idea of the circulation of the blood could not be developed until the idea of a pump was known and somewhat understood. The behaviour of a real system cannot be developed until one can see simpler models within it. But what are these simpler models? They are generally among the large variety of models that have been created in the past, often recreational or fanciful.

Perhaps the clearest example is graph theory, where Euler made a simple model of the reality that he was studying, then later workers found that model useful in other situations. Graphs were then recognised as present in many early problems: river crossing in ca. 800, knight's tours in ca. 900, etc.

Thus, *recreational mathematics helps as a major source of mathematical models, techniques, and methods*, which are the raw material for mathematical research, in the same way that mathematics in general serves as a source of models for the physical world. I think this is the explanation of the utility of recreations in mathematics and the utility of mathematics in the real world.

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The map of Königsberg would appear to be in the first *Anhang* (appendix).

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