MATHEMATICAL GAMES, ABSTRACT GAMES

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The board diagrams in this book were made with the PostScript language from Adobe Systems and derived from the original set of board-design functions implemented by Cameron Browne.

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To my family
JPN

To Iguana, unreal llama
JNS

Contents

Chapter 1

The World of Games

Playing is an activity as old as civilization. There are two usual meanings for the word *play*: any childish activity without rules, and game playing, where rules are essential. This book is about games.

Some games are thousands of years old; they were probably the first strictly mental activities created by man. Nowadays, we learn about old games mainly when they are related to recreational mathematics. An example is Mancala, which uses a board reminiscent of an abacus, an old calculation device.

It is natural to classify games according to their rules. This book does not deal with games where chance plays a role (with the use of dice, for instance), or where there is some hidden information that a player has and his adversary does not (like most card games). Games that avoid these two situations are usually called abstract games.

We do not intend to write down the rules of chess or checkers. Our goal is to present a set of games, some not yet ten years old, that can help the reader and his family and friends find a leisure activity, especially those who enjoy intellectual challenges. Almost all the games treated here (the main exceptions are Go and Hex) have not yet been explored, but were chosen because the authors found in them strategical or tactical qualities, enough to offer hours of ludic pleasure.

We tried to produce a visually appealing and user friendly text, inviting the readers to play abstract games. Some more serious analysis is encouraged, for the enthusiasts.

When learning a game, it is strongly suggested that board and pieces be brought in place and the moves played out. Otherwise, it is easy to misunderstand some part of the rules or other aspects of the games. The authors are available for any clarification, mainly by e-mail.

This book is divided in the following parts:

1) This chapter, which consists of an introduction, an historical survey, and a superficial description of the actual panorama of board games.

2) A section dedicated to games for two players. For each of them, we describe the rules, the necessary material, and make some strategical and tactical commentaries on some positions.

3) A full chapter is occupied with the introduction of a class of mathematical games, which can be, at least in principle, completely analyzed in a quick and efficient manner. Nim and games on graphs are two such examples.

4) A section on games for three players. We present a set of games fit for the occasion when three people want to play and nobody wants to be left out. When the number of players is larger than two, new social complexities

arise, such as alliances, diplomacy, threats, and bluffs. A good game for these situations should be interesting without promoting personality clashes or extra-game arguments.

5) A glossary, to help minimize the ambiguity of the rules. It also explains the notations we use throughout the book for boards, positions and moves.

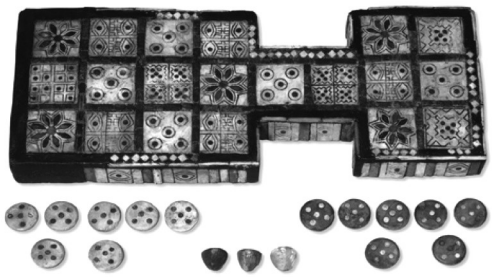
Brief Historical Journey

The expression "mathematical game" can be used to refer to a game, to a puzzle or to a problem of any degree of difficulty. The history of mathematics shows that mathematicians of all ages dedicated some of their energies to activities that could be classified as games. Some fields of mathematics were born this way.

All civilizations produce and play games. We do not know why, but their cultural and educational relevance is clear.

The games we'll be treating in this book are usually called *abstract* or *mathematical* games, but sometimes also *games of strategy*. We need some boards, that we'll describe, but in some cases pencil and paper, or piles of beans, are enough to play them.

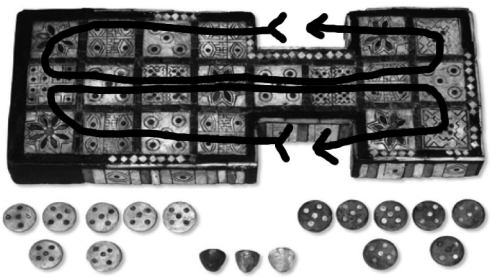
The oldest game for which we know the rules is the Royal Game of Ur, from Babylon. It was found in the 1920s, and it is just a race between two opponents.



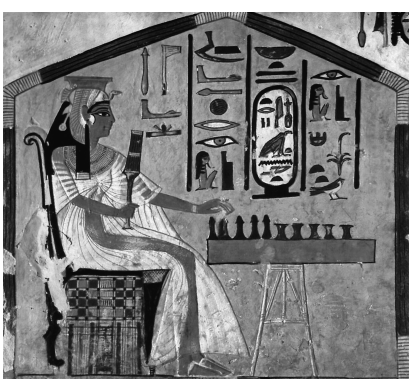
Two sets of pieces and a set of tetrahedrical dice can be seen here.

We think that the pieces used to move in the directions illustrated on the next page, being the length of each dispacement decided by the throwing of the dice.

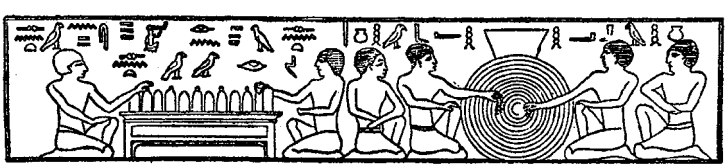
The first player to finish his course would be the winner.



In ancient Egypt's tombs, as in the *Book of Dead*, we can find references to game playing, as can be seen in the picture below, from an inscription found in Nefertari's tomb:

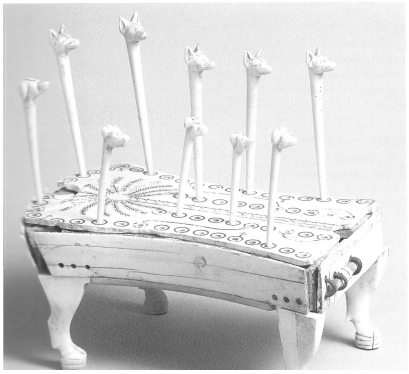


or in this representation of players:



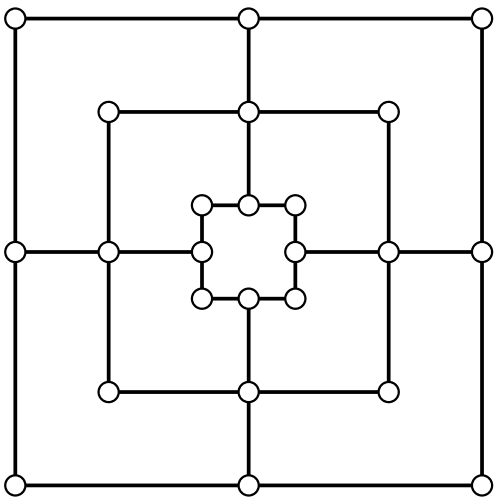
The two games depicted here are Senet (on the left) and Mehen (on the right, which has been identified with a representation of the god of games, the serpent god). Senet, which is also a race game, has a strong religious relevance. In the game represented above it looks like there is only one player. However, it is life after death that is being gambled on against the god of the afterlife.

Another race game from ancient Egypt was Dogs and Jackals:

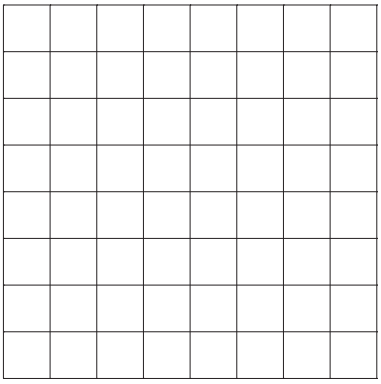


Once again we find almost independent courses. The players do not interact much during play. The lines connecting some pairs of holes (one "good" and one "bad"), to add drama to the game, are the first examples of such devices, well known in the popular Chutes & Ladders.

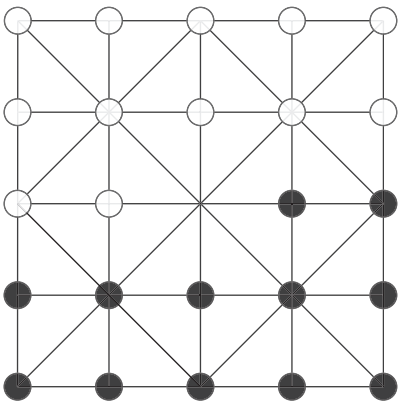
Nine Men Morris is a typical example of a family of alignment games. Probably known already in ancient times, only very recently (1996), through a very impressive computer analysis, it was found that, with perfect play, the game is a draw. It was very popular in western Europe in the Middle Ages.



Ludus Latrunculorum, the soldier game, was a favorite of the Roman army. It is a military strategy game, the board representing a battlefield, the pieces representing soldiers. We are not sure about the original rules. The board was rectangular with variable dimensions. Some were carved in stone and can still be found in several Roman monuments.

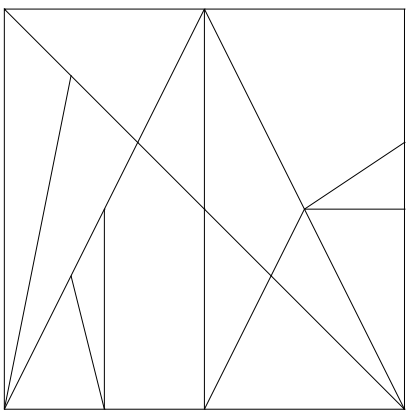


Alquerque, the grandfather of checkers, is also very old. We can find its vestiges in ancient monuments, raising difficult questions about its real age.

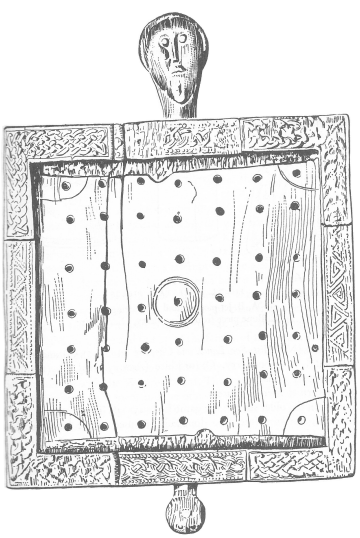


Archimedes (287-212 B.C.) described a geometric puzzle, the stomachion. It is a game similar to the tangram (a Chinese puzzle, still popular in the West), made up by fourteen plane shapes that can be assembled to make a square.

In the *Sand Reckoner*, Archimedes is believed to have described the combinatorial properties of stomachion, but that description has not survived.

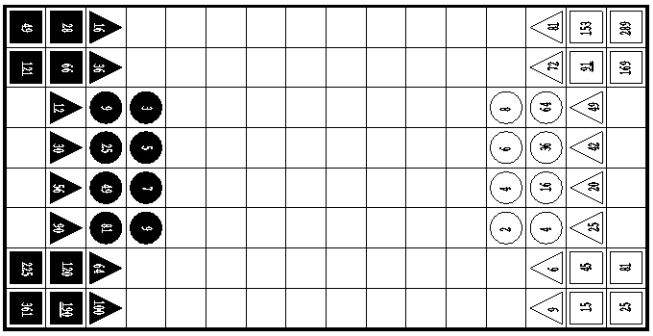


In the Middle Ages many board games were used. From the tenth century Ireland, the board of Fithcheall:



In this game, one of the two players takes the central square with his king, surrounding it with less valuable pieces. To win, this player needs to take his king to the periphery of the board. His adversary has more pieces, but no king. To win, this player needs to kill the king. The movements were orthogonal, like Chess rooks, and only custodian captures were permitted, surrounding the victim (see Glossary).

In the Middle Ages the erudite classes used to play several games. Some circulated only among universities, churches, and other educated places, where people could understand the complexity of the rules. One such example was Rithmomachia, also known as the Philosopher's Game, which was associated with the teaching of Boethius's *Arithmetic* for five hundred years.

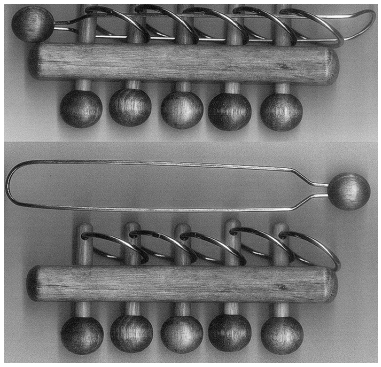


Rithmomachia was a pedagogical game, especially designed to grasp some numerical relations, such as progressions.

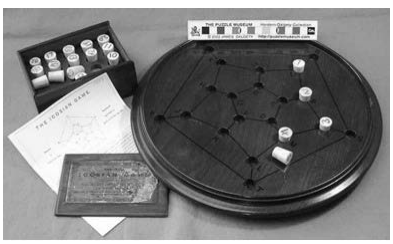
Besides its practical utility, arithmetical knowledge had religious and moral value. Accordingly, Rithmomachia had an important role in the education of the learned classes for a few centuries.

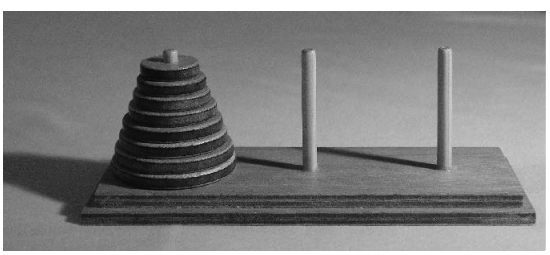
The movements depended on the shape of the pieces, and captures depended also on the numbers each piece displayed. Victory was accomplished by occupying some squares in the adversary's half-board, with the corresponding numbers in special progressions.

Luca Pacioli, one of the leading mathematicians of the 15th century, was also a recreational mathematics fan. He wrote what is considered to be the first book on this subject. One of the puzzles described there is the now famous Chinese Rings:

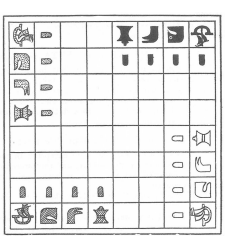


The Irish mathematician Sir William Rowan Hamilton, in 1857, created a game, Icosian, which was a commercial flop. However, it was related with a very important concept in graph theory, *Hamiltonian circuits*.





One of the oldest games from which chess developed is Chaturanga, a popular game in India during the 6th century (played by four people).



Chess rules have varied a lot since then. In the Middle Ages, piece movement was sometimes decided by the use of dice. From the 16th century on, chess became more popular than Rithmomachia among educated classes in Europe. Today chess is played by millions, and the Philosopher's Game is history.

Mathematical games are usually a favorite subject in books focused on the popularization of mathematics, and even in some pedagogical texts. However, math here is always apparent. In the games we call abstract the situation is different, the mathematical content is hidden. Playing the game is, in an abstract way, doing mathematics with the ludic side of each activity dominant.

G. H. Hardy, one of the most important mathematicians of the 20th century, said that the only difference between a chess problem and a mathematical theorem lies in their relevance. Abstract games and pure mathematics are the same... .

We believe that the practice of good games nurtures the intellect. We do not know how this mechanism actually works, but we believe that some good comes from playing interesting board games. Below we'll try to pinpoint the qualities that make a game worth playing.

The State of the Art

In some human activities, as in science and philosophy, today's production has attained such levels as to lead us to consider that we live in a golden age. The same is true for the market of games. In the past 25 years, board games have been invented at a pace never matched previously. In spite of the advent of computer games, new board games appear in quantity and quality, mainly in Germany, France and the USA. It is difficult, of course, to find a commercial success like Monopoly or Risk, but the German game from 1994, The Settlers of Catan, sold millions and is still selling well.

If we focus on abstract games, we find a different reality. There is some serious production of board games — Gigamic and Kris Burm's Gipf Project are good examples — but the Internet seems to be the natural habitat for this kind of game. Here, the communication among people sharing the same interests, but geographically apart, is easy. The emergence of dedicated forums, and other virtual platforms, to share information has triggered an outburst of creativity. Abstract games are particularly well adapted to this means of communication: it is easy to describe the rules and the boards, using only keyboard characters. It became possible to show a new game immediately after its conception to an interested public. In a few days a game can be tried, improved, or dismissed. This new version of postal chess, in which a turn could take weeks, is now at our fingertips. New servers were created to administer thousands of games.

Among traditional abstract games the best known are chess, with its regional dialects, Go, several Mancala games, Alquerque, and checkers.

During the 19th century and the first half of the 20th some modern classics were born, like Othello/Reversi, Chinese Checkers, Four-in-a-Row and Renju (essentially a five in-a-row with a complex protocol of initial rules). In our times, hundreds of new games appear during each decade. Most of them are just variations on old games, but others contain new ideas and interesting concepts.

Computer science plays a special role in the development of new games. It is not difficult to implement an algorithm that emulates a player. Some of the best chess programs play already at the grandmaster level. A landmark was Zillions of Games, which appeared in 1998. This application can implement a large quantity of different games (see the Electronic References).

Even without special software, it is possible to find someone on the Internet willing to try a new game. A few explorations later, it becomes clear whether the game has potential or if, on the contrary, it does not have any ludic qualities. When strong and weak points of a game are spotted, it is possible to improve the game, or to abandon it. The overall quality of board games increases this way. Possibly, if chess, Go, or checkers were invented today, they may not have the success they know. Other games, like the ones presented here, if created under different circunstances, could have been very successful. We hope that this book can help keep alive some of these games. They deserve to be passed on, if not for their history, then for their intrinsic value.

Families of Games

It is natural to classify games into families according to victory goal. If a game has more than one way of winning, the game belongs to more than one family naturally. We identify the following:

Territory games — A player tries to get as much area as he can (the calculation of the area depends on the rules of the particular game). In this book we present Go, Anchor, and Dispatch.

Blocking games — The winner is the player that blocks the adversary from any legal move. We include Amazons, Iqishiqi, Pawnographic chess, Campaign, Hobbes, UN, and all the Nim games.

Capture Games — The winner is the first to capture a set of pieces. In this book: Annuvin, Gogol, Nosferatu, and Hobbes.

Positional games — Victory depends on disposing one or more pieces in a certain part of the board. We have Aboyne, Pawnographic chess, Epaminondas, Gogol, Iqishiqi, and Slimetrail.

Pattern games — A pattern, usually a line, must be formed with the

pieces. Examples: Gomoku, Havannah, Intersections, Semaphore, SanQi, and Stooges.

Connection games — The winner is the player that first forms a group of pieces satisfying some condition (for instance, connecting two sides of the board). This book presents a wide variety of connection games: Hex, Y, Nex, Gonnect, Jade, Havannah, and Lines of Action.

Inventing a new game is not easy, but the creation of a family has a different level of difficulty; here we need a completely new concept, not only a new set of rules. Besides, this new concept should be able to generate several good original games.

Variants

Everybody familiar with the game of checkers knows the *giveaway* variant, i.e., with the usual rules (capture is mandatory), the winner is the first player unable to move. Most of the traditional games have variants or other ways of playing, not as exact as the "official rules." Some of these variants have a regional character. For example Portuguese checkers and French checkers are different games, and English checkers is still another. Chess also has regional variants (actually, the international rules of FIDE are nothing more than a regional variant that went global), like Xiang-Qi (in China) and Shogi (in Japan). When there are several regional variants of the same game, the expression families of games is used (not with the previous meaning). Some examples: the family of chess, the family of checkers, and the family of Mancala (with hundreds of variants, mainly from Africa).

Traditional variants only account for a small part of the total. Most of them are modern. A good example is chess, which has thousands of variants, including many good games. Some of these were created by leading professional players, such as Laska (a checkers variant by Emanuel Lasker) and Fischer chess (a chess variant by Bobby Fischer).

How does one create a variant? Quite often a variant appears due to a lack of clear communication of the original rules. Probably, a new traditional version was created when a traveller was explained a game in a foreign language. Other variants were born due to deficient interpretation of written rules (which could happen with this book). But most variants were made on purpose. When a game is analyzed, played, and its strategy and tactics understood, then some changes start making sense. When the changes work well and give rise to an appealing game, a new variant is born.

The way new variants arise depends a lot on the particular game. But there are some general principles. There are meta-rules (also known as mu-

tators) that change consistently a big class of games. For instance, the progression meta-rule, according to which the first player plays once, the second twice, the first player now plays three times, and so on. Usually this meta-rule is excessive, and has to be softened (see an example in Y). Applied to chess, with the restriction that "a sequence of moves ends before its natural length if a check occurs" gives rise to the well known variation progressive chess.

Here we present a list of other meta-rules:

a) Dagger (see Glossary): One of the players can play twice in one turn (see Gomoku, p. 62).

b) Introduction of neutral pieces (pieces that do not belong to either player), with the convention that each player can change two neutral pieces for two of his own and one of his by a neutral (see Nex, p. 98).

c) Pocket: Each player can pass his turn, saving that turn for a later moment (as if he had bought a dagger, the price being a pass).

d) Restrictions: To build an initial handicap to account for the unbalanced levels of the players. For example, in chess, the strongest player could start without the Queen's Knight and play White.

e) Pie rule (see Glossary): This could be the most important meta-rule. The first player does n moves (with both colors) and his adversary chooses the color he wants for the rest of the game. This rule makes it possible to balance some games where the first player has, without the pie rule, great advantage, such as Gomoku. This rule is assumed in several games in this book.

We can even combine meta-rules, if the particular game is a good fit for that. For example, progressive chess with a pie rule with $n = 2$ is a good possibility.

Quality control

How can we tell a good game from a bad one? How can we assess the quality of an abstract game? Which properties would we like to find in a game so we would spend time with it? There are no final answers to these questions, but we can point to some factors.

One of the most important elements is *depth*, or strategic complexity. How specialized can one be in one game? For example, tic-tac-toe has a very low complexity; it is easy to master its strategic subtleties and come out with a plan that leads consistently to draw games. On the opposite side is chess; there are several levels of sophistication at which chess can be played, from beginner to grandmaster.

Consider the following definition: A player X is one level above player Y if X beats regularly Y two out of three times. In chess, this accounts for a difference of about 100 ELO points (ELO is a system designed to assess the relative strength of the players.) If the beginner level corresponds to about 1,000 points and the best player ever has a level of 2,900, there are 19 levels of playing the game (actually a little more is true, since for stronger players the ELO classification does not grow so fast).

Tic-tac-toe may not have more than three levels (if we assume that children do not know the best strategy). Checkers is between tic-tac-toe and chess. Go (see p. 51), the old traditional game, has more than 30 levels of specialization. Check Thompson's article [THO] for an early discussion on these matters.

Clarity is another important issue. Clarity tries to answer the question "How difficult is it to create a good tactic or strategy?" The easier it is to visualize moves in the future, the greater clarity a game has. If a game has little clarity it becomes difficult to consider possible moves and anticipate the adversary's threats. It is nice when our victory is due to wise planning, not to a blunder of the opponent. Probably the clearest game in this book is Hex, and a very obscure one is Lines of Action.

Another important property is *drama*. A game has drama if it is possible to overcome a difficult situation by surprising strategic or tactical moves (for instance, sacrifices.) Chess is a great example of a dramatic game, as we can see in the problems in the literature. If a game between a weak player and stronger one is interrupted, and the weaker player's position given to a master, the game can still be interesting. Very deep games, like Go, usually have drama. But, on the other hand, a good game should be *decisive*: there must exist a situation towards which a player conducts the game that ensures him of the victory, independent of the level of his adversary. A game without this characteristic becomes confusing, producing cyclic dramatic events, with no end.

The average *time* a game takes is also important. A chess game takes about 40 moves. One of checkers slightly less. Tic-tac-toe takes nine moves, at most. A game of Go can, after a 100 moves, be undecided. Of course it all depends on the free time each of us has, but nowadays games that take too long are penalized. (if Go were invented now, it would have a hard time surviving.) Games like tic-tac-toe, Hex, and Amazons, in which the cells of the boards are successively occupied, cannot go on too long, but a chess game could last, theoretically, over a thousand turns.

Ramification, that is, the number of possible moves a player can play, on average, in each move. It is in some ways the opposite of clarity. In

principle, the more possible moves there are, the less clear a game is. This property is important to computer scientists. The larger the ramification, the harder it is to design good software for a game. For example, chess's average ramification is about 40 moves, chess programs play very well. Go has an average ramification of 180 moves, the best computer programs play at beginner's level. Some of the games we present here have a ramification in the thousands, without penalizing clarity, as Nex or the progressive variant of Y.

The *interaction* is also important. This property addresses the level at which the pieces act on each other (a property introduced by Cameron Browne in [BRO]). Games with low interaction are just "individual race" games played at the same time, such as Chinese Checkers and Halma. In games with good interaction, it is possible to create complex configurations with the adversary's pieces, improving the quality of the game and increasing the number of relevant tactical moves. The use of neutral pieces plays a role here; see Sanqi, Nex, Iqishiqi and Hobbes.

We tried to include games that classify well according to these criteria. Most of them are very recent and almost unknown. Some are the result of our personal experience. We hope the readers enjoy the games.

Electronic References

The World of Abstract Games: www.di.fc.ul.pt/~jpn/gv A website of one of the authors, containing hundreds of board games from all ages and places. It is the main reference for the games for two players in this book.

Chess variants: www.chessvariants.com The ultimate website on chess variants.

Zillions of Games: www.zillionsofgames.com A commercial software package (there is a free demo version), that implements over a thousand board games, with an open specific language for the creation of new games.

Games by e-mail: www.gamerz.net/pbmserv/ An electronic games server created and administered by Richard Rognlie, where you can find opponents for dozens of games.

Board construction: http://www.di.fc.ul.pt/~jpn/gv/dobpt.htm Lots of boards used by the games described here are not easy to get. It is possible to print hard copies of boards and use chess or checker pieces or other materials to play. In this page we make available several PDF files that everyone can print. This page also contains hints and ideas to build physical game sets.

Associação Ludus: http://ludicum.org A Portuguese organization focused on Recreational Mathematics, including Mathematical Games.

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