The concept of number

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Each individual number is an independent object

55. Having recognized that a statement of number is an assertion about a concept, we can attempt to supplement the leibnizian definitions of the individual numbers by means of the definitions of 0 and of 1.

Right away we might say: the number 0 applies to a concept, if no object falls under that concept. Here, however, "no" appears to have been substituted for 0, with which it is synonymous. Therefore the following wording is preferable: the number 0 applies to a concept if, no matter what *a* might be, the statement always holds that *a* does not fall under this concept.

Similarly we could say: the number 1 applies to a concept *F* if it is not the case that no matter what *a* is, *a* does not fall under *F*, and if from the statement

'*a* falls under *F*' and '*b* falls under *F*'

it always follows that *a* and *b* are the same.

We must still define in general the transition from one number to the next. We will try the following formulation: the number $(n+1)$ applies to the concept *F* if there is an object *a* which falls under *F* and such that the number *n* applies to the concept "falling under *F* but not [identical with] *a*."

56. These definitions appear so natural, following our previous results, that an explanation is called for to show why they cannot satisfy us.

The last definition will most quickly arouse hesitation, for, strictly speaking, the sense of the expression 'the number *n* applies to the concept *G*' is just as unknown to us as that of the expression 'the number $(n+1)$ applies to the concept *F*'. To be sure, we can say by means of this and the next-to-last definition what

'the number $1+1$ applies to the concept *F*'

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means, and then, using this, indicate the sense of the expression

'the number $1+1+1$ applies to the concept *F*', etc.

But, to give a crude example, we can never decide by means of our definitions, whether the number *Julius Caesar* applies to a concept, whether this well-known conqueror of Gaul is a number or not. Furthermore, we cannot prove with the help of our attempted definitions that *a* must equal *b* if *a* applies to the concept *F* and *b* applies to the same concept. The expression '*the* number which applies to the concept *F*' would, therefore, not be justifiable, and it would consequently be completely impossible to prove a numerical equality because we could never isolate a definite number. It is only apparent that we have defined 0 and 1; as a matter of fact, we have only determined the sense of the expressions

'the number 0 applies to'

and

'the number 1 applies to';

but it is not permissible to isolate in these 0 and 1 as independent, recognizable objects.

57. Here is the place to examine somewhat more closely our statement that a statement of number involves an assertion about a concept. In the sentence 'the number 0 applies to the concept *F*', 0 is only a part of the predicate, if we consider the concept *F* as the actual subject. Therefore I have avoided calling numbers like 0, 1, 2 properties of concepts. The individual number appears as a separate independent object for the very reason that it forms only a part of the assertion. I have already called attention above to the fact that we say 'the [number] 1' and, by means of the definite article, set up 1 as an object.

This independence appears everywhere in arithmetic, e.g., in the equation '$1+1=2$'. Since the important thing here is to grasp the concept of number in such a way that it is useful for science, it needn't disturb us that in everyday usage the number appears attributively. This may always be avoided. E.g., the sentence 'Jupiter has four moons' may be rearranged to form 'The number of Jupiter's moons is four'. Here the 'is' is not to be considered merely a copula, as in the sentence 'the sky is blue'. This is shown by the fact that one can say 'the number of Jupiter's moons is four' or 'is the number four'. Here 'is' has the sense of 'is equal to', 'is the same as'. We have, therefore, an equation which asserts that the expression 'the number of Jupiter's moons' denotes the same object as the word 'four'. And the form of the equation is the reigning one in arithmetic. The fact that nothing about Jupiter or about a moon is contained in the word 'four' is no objection to this interpretation. Neither is there anything in the name 'Columbus' to suggest discovery or America, and nonetheless the same man is called both Columbus and the discoverer of America.

58. One could object that we cannot at all represent[1] to ourselves the object which we call four or the number of Jupiter's moons as something separate and indepenent. However, it is not the separateness which we have given the number that is at fault. To be sure, one would like to believe that in picturing the four spots of a die something appears which corresponds to the word 'four' - but that is an illusion. Think of a green meadow and see whether the picture changes when the indefinite article is replaced by the number 'one'. Nothing is added, but there is certainly something in the picture corresponding to the word 'green'.

If one pictures for himself the printed word 'gold', one will not at first associate any number with it. Were one now to ask himself how many letters the word has, the result would be the number 4; the picture, however, will be in no way more definite, but can remain wholly unchanged. The added concept "letter of the word 'gold'" is the very thing in which we discover the number. In the case of the four spots of a die the situation is somewhat less obvious because the concept is forced upon us so directly by the similarity of the spots that we hardly notice its intrusion. The number can be *pictured* [translator's italics] neither as a separate object nor as a property of an outward thing, because it is neither something sensible nor the property of an outward thing. The situation is probably most clear in the sense of the number 0. One will try in vain to picture 0 visible stars. To be sure, one can think of the sky completely covered up by clouds; but there is nothing in this picture which might correspond to the word 'star' or to the 0. One is only imagining a situation in which one may conclude: now no star may be seen.

59. Perhaps each word awakens some sort of picture for us, even a word like 'only'. The picture, however, need not correspond to the content of the word; it can be an entirely different one for different men. One will then probably imagine a situation which evokes a sentence in which the word occurs; or the spoken word might call forth the written word in one's memory.

This does not occur only in the case of particles. There can be no doubt that we lack any idea [picture] of our distance from the sun. For, even if

[1]In the sense of 'picture'.

we know the rule about the number of times we must multiply a unit of measure, nevertheless any attempt by this rule to sketch a picture which even slightly approaches the one desired is doomed to fail. This is, however, no reason to doubt the correctness of the computation by which the distance has been found, and it in no way hinders us in basing further conclusions on this being the distance.

60. Even such a concrete thing as the earth we cannot picture in the way that we have learned it actually to be, but rather we are satisfied with a sphere of medium size, which serves us as a symbol for the earth, knowing nevertheless that the two are very different from one another. Now although our picture often does not at all meet the requirements, still we make judgments with great certainty about an object like the earth, even where its size is concerned.

Thought often leads us far beyond the imaginable without thereby depriving us of the basis for our conclusions. Even if, as it appears, thought without mental pictures is impossible for us men, still their connection with the object of thought can be wholly superficial, arbitrary, and conventional.

The unimaginability of the content of a word is no reason, then, to deny it any meaning or to exclude it from usage. That we are nevertheless inclined to do so is probably owing to the fact that we consider words individually and ask about their meaning [in isolation], for which we then adopt a mental picture. Thus a word for which we are lacking a corresponding inner picture will seem to have no content. However, we must always consider a complete sentence. Only in [the context of] the latter do the words really have a meaning. The inner pictures which somehow sway before us (in reading the sentence) need not correspond to the logical components of the judgment. It is enough if the sentence as a whole has a sense; by means of this its parts also receive their content.

This observation seems to me to be useful in throwing light on several difficult concepts, such as that of the infinitesimal,[2] and its scope is probably not limited to mathematics.

The separateness [independence] which I require for the number is not intended to mean that a number-word used outside of the context of a sentence shall denote anything, but rather I want only to exclude its use as a predicate or attribute, for such a use somewhat alters its meaning.

[2]What is in question here is defining the sense of an equation like

$$df(x)=g(x)dx$$

rather than finding an interval bounded by two distinct points and of length dx.

61. But, one might object, even if the earth is really unimaginable, still it is an external thing having a definite place. Where, however, is the number 4? It is neither outside of us nor inside of us. Taken in spatial terms, this is correct. A determination of the place of the number 4 makes no sense. But, from this it follows only that the number 4 is not a spatial object, not that it is no object at all. Not every object is somewhere. Even our mental pictures[3] are in this sense not in us (subcutaneously). In us there are ganglia cells, blood particles, etc., but no mental pictures. Spatial predicates are not applicable to them: the one is neither right nor left of the other. Mental pictures have no distances between them which may be stated in millimeters. When nevertheless we refer to them as in us, we mean that they are subjective.

Even if the subjective has no spatial location, however, how is it possible for the number 4, which is objective, to be nowhere? Now I maintain that there is no contradiction here. The number 4 is, as a matter of fact, exactly the same for everyone who works with it; but this has nothing to do with spatiality. Not every objective object has a place.

In order to obtain concept of number, one must determine the sense of a numerical equation

62. How shall we have a number, then, if we can have no idea or picture of it? Only in the context of a sentence do words have meaning. We must, therefore, define the sense of a sentence in which a number-word occurs. This seems at first to leave a lot of latitude, but we have already determined that number-words are to be understood as standing for independent objects. This already specifies a class of sentences which must have a sense, the class of those sentences which express the recognition [of a number as the same number]. If for us the symbol *a* is to denote an object, then we must have a criterion which determines in every case whether *b* is the same as *a*, even if it is not always within our power to apply this criterion. In our present case, we must explain the sense of the statement:

> 'the number which applies to the concept *F* is the same number as that which applies to the concept *G*',

i.e., we must reproduce the content of this statement in another way without using the expression

> 'the number which applies to the concept *F*'.

In doing this, we give a general criterion for the equality of numbers.

[3] This word is understood purely psychologically, not psychophysically.

Once we have obtained such a means of grasping a definite number and recognizing it as such, we can assign it a number-word as its proper name.

63. *Hume* (Baumann 1868–9, 2: 565) has already mentioned such a means: "If two numbers are so combined that the one always has a unit which corresponds to each unit of the other, then we claim they are equal." In more recent times, the opinion seems to have found much sympathy among mathematicians, that the equality of numbers must be defined in terms of a one-to-one correspondence. Immediately, however, there arise certain logical hesitations and difficulties, which we must not pass by without examination.

The relationship of equality does not hold only among numbers. It seems to follow from this that the relationship should be defined especially for numbers. One would think it possible to derive a criterion of when numbers are identical with one another from a previously determined concept of identity together with the concept of number, without its being necessary, for this purpose, to define a special concept of numerical identity.

Contrary to this, it should be noted that, for us, the concept of number has not yet been defined, but rather is to be determined by means of our definition of numerical identity. We intend to reconstruct the content of judgments interpretable as expressing identities each side of which is a number. We do not, therefore, want to define equality especially for this instance, but we wish rather, by means of the already familiar concept of equality, to determine that which is to be considered equal. This seems indeed to be a very unusual type of definition, which has probably not yet received sufficient attention from the logicians. Nevertheless, that it is not entirely unheard of may be shown by a few examples:

64. The judgment: 'the [straight] line *a* is parallel to the [straight] line *b*', or, symbolically:

$$a \parallel b,$$

can be interpreted as an equation. If we do this, we obtain the concept of direction and say: 'the direction of line *a* is the same as the direction of line *b*'. Hence, we replace the symbol '$\parallel$' by the more general '=', by distributing the particular content of the former to *a* and *b*. We split up the content in some way other than the original way and thus obtain a new concept. Often the situation is interpreted conversely, and several teachers define: parallel lines are those having the same direction. The theorem "if two straight lines are parallel to a third, then they are parallel to one another" can then be very easily proved on the basis of the similarly worded equality theorem. Unfortunately, this method reverses the natural order of things. For everything geometric must indeed be intuitive, at least originally. Now I ask whether anyone has ever had an intuition of the direction of a straight line? Of the straight line, yes, but can one also distinguish intuitively this line from its direction? Rather difficult! This concept is found only by means of a mental activity connected with intuition. On the other hand, one has a picture of parallel lines. That proof comes about only through a trick in which what is to be proved is covertly presupposed in the use of the word 'direction'; for, were the statement: 'if two straight lines are parallel to a third, then they are parallel to one another' false, then one could not change '$a \parallel b$' into an equation.

Thus one can obtain from the parallelism of planes a concept which corresponds to that of direction among straight lines. I have seen the name 'orientation' used for this concept. From geometric similarity there arises the concept of shape, so that, e.g., instead of 'the two triangles are similar', one says: 'the two triangles have the same shape' or 'the shape of the one triangle is equal to the shape of the other.' Similarly one can also obtain from the collinear relationship of geometric figures a concept for which a name is probably still lacking.

65. Now, in order to move, e.g., from parallelism[4] to the concept of direction, let us try the following definition: the sentence

'line *a* is parallel to line *b*'

is to be synonymous with

'the direction of line *a* is the same as the direction of line *b*'.

This definition departs from common practice insofar as it apparently defines the already familiar relation of equality, while it should in actuality introduce the expression 'the direction of line *a*', which occurs only incidentally. From this there arises a second hesitation; viz., whether, through such a stipulation, we could not become involved in contradictions with the familiar laws of equality. What are these? They will be developed as analytic truths from the concept itself. Now, Leibniz defines:[5]

[4]In order to be able to express myself more comfortably and to be more easily understood, I speak here of parallelism. The essential parts of these discussions are very easily carried over to the case of numerical equality.

[5]*Non inelegans specimen demonstrandi in abstractis* (Erdmann 1840: 94).

"Eadem sunt, quorum unum potest substitui alteri salva veritate." ["Things are equal which may be substituted for one another without change of truth [value]."]

I will adopt this definition. Whether, like Leibniz, one says 'the same' or 'equal', is of little import. 'The same' does seem to express complete agreement, 'equal' only agreement in this respect or that. One can, however, assume a manner of speaking in which this difference is eliminated, e.g., by saying instead of 'the lines are equal in length' that 'the length of the lines is equal' or 'the same'; instead of saying 'the surfaces are equal in color' one might say 'the color of the surfaces is equal [identical]'.

And this is the way we used the word in the foregoing examples. In fact, all the laws of equality are contained in the principle of universal substitutivity.

In order to justify our proposed definition of the direction of a straight line, we would have to show, then, that

'the direction of *a*'

can be everywhere replaced by

'the direction of *b*',

if line *a* is parallel to line *b*. This is simplified by the fact that, at first, we know no assertion about the direction of a straight line other than its agreement with the direction of another straight line. We would therefore need to demonstrate only the substitutivity in such an equation or in contexts which would contain such equations as component parts.[6] All other statements about directions would have to be defined first, and for these definitions we can adopt the rule that the substitutivity of the direction of a straight line for that of one parallel to it must be preserved.

66. Still a third hesitation arises, however, concerning our proposed definition. In the sentence

'the direction of *a* is equal to the direction of *b*',

the direction of *a* appears as an object,[7] and we have in our definition a means of recognizing this object, should it appear in some other guise,

[6]For example, in a hypothetical judgment an equality of directions could occur either as antecedent or as consequent.

[7]The definite article points to this. A concept is for me a possible predicate in a singular thought content, an object a possible subject of the latter. [Although the terminology of "thought contents" has been adopted, Frege must not be taken to mean anything psychological by 'thought'. For Frege a "thought content" is what is asserted in a statement, asked in a question, etc....] If, in the sentence 'the direction of the axis of the telescope is equal to the direction of the earth's axis', we consider the direction of the telescope's axis to

such as the direction of *b*. However, this method is not sufficient for all cases. One cannot use it to decide whether England is the same as the direction of the earth's axis. Please excuse this apparently nonsensical example! Naturally, no one is going to confuse England with the direction of the earth's axis; but this is not owing to our definition. The latter says nothing about whether the statement

> 'the direction of *a* is equal to *q*'

is to be affirmed or denied, if *q* itself is not given in the form 'the direction of *b*'. We lack the concept of direction; for, if we had this, then we could stipulate that, if *q* is not a direction, then our statement is to be denied; if *q* is a direction, then the earlier definition decides. It is now but a step away to define:

> *q* is a direction if there is a straight line *b* whose direction is *q*.

However, it is clear that we have now come around in a circle. In order to apply this definition, we would first have to know in each case whether the statement

> '*q* is equal to the direction of *b*'

was to be affirmed or denied.

67. If we were to say: *q* is a direction if it is introduced by means of the foregoing definitions, then we would be treating the manner by which the object *q* is introduced as a property of it, which it is not. The definition of an object, as such, really says nothing about that object; rather it stipulates the meaning of a symbol. Once that has happened, the definition becomes a judgment which treats of the object: it now no longer introduces the object but stands on equal footing with other statements about it. To choose this way out is to presuppose that an object could be given in one way only; otherwise it would not follow from the fact that *q* is not introduced by means of our definition that it could not be so introduced. The import of any equation would then be that what is given us in the same way should be recognized as the same. But this principle is so obvious and so unfruitful that there is little to be gained by stating it. As a matter of fact, no conclusion could be drawn from it which would not be the same as some premise. The many-sided and broad applicability of equations is based rather on the fact that something is recognizable again even though it is given in a different way.

be the subject, then the predicate is 'equal to the direction of the earth's axis'. This is a concept. But the direction of the earth's axis is only a part of the predicate; the direction is an object, since it can also be made the subject.

68. Since this method fails to yield a sharply delimited concept of direction and, for the same reason, would yield no such concept of number, let us try a different tack. If line *a* is parallel to line *b*, then the extension of the concept "line parallel to line *a*" is the same as the extension of the concept "line parallel to line *b*"; and conversely: if the extensions of the aforementioned concepts are equal, then *a* is parallel to *b*. Let us try, then, to define:

> the direction of line *a* is the extension of the concept "parallel to line *a*"
>
> the shape of triangle *d* is the extension of the concept "similar to triangle *d*."

If we want to apply this to our case, then we must substitute concepts for the lines or the triangles and, for parallelism or similarity, the possibility of correlating in one-to-one fashion the objects falling under the one concept with those falling under the other. As an abbreviation, I will call the concept *F* equinumerous[8] with the concept *G*, if this possibility exists; I must, however, request that this word be considered an arbitrarily chosen notational device whose meaning is not to be taken from its linguistic composition, but rather from the foregoing definition.

I define accordingly:

> the number which applies to the concept *F* is the extension[9] of the concept "equinumerous with the concept *F*."

69. That this definition is correct will, at first perhaps, not be so clear. Don't we mean something other than [different from] a number by the extension of a concept? What we do mean becomes clear from the basic statements that can be made about extensions of concepts. They are the following:

[8][Frege coined 'gleichzählig' for this. In his translation, J. L. Austin (Frege 1950) uses 'equal' and adds the following footnote: "*Gleichzählig* – an invented word, literally 'identinumerate' or 'tautarithmic'; but these are too clumsy for constant use. Other translators have used 'equinumerous'; 'equinumerate' would be better. Later writers have used 'similar' in this connection (but as a predicate of 'class' not of 'concept')." – Tr.]

[9]I think we could say for 'extension of the concept' simply 'concept'. However, there might be two objections:

1. This stands in contradiction to my earlier assertion that the individual number is an object, the latter being indicated by the use of the article in expressions like "the 2," by the impossibility of speaking about ones, twos, etc. in the plural, and by the fact that the number makes up only a part of the predicate of a statement of number.

2. Concepts can have the same extension without coinciding.

Now I am of the opinion that both these objections can be met, but doing this would lead us too far astray. I presuppose that one knows what the extension of a concept is.

1. that they are equal,
2. that the one encompasses more than the other.

Now the statement

> 'the extension of the concept "equinumerous with the concept *F*" is the same as the extension of the concept "equinumerous with the concept *G*"'

is true if and only if the statement

> 'the same number applies to the concept *F* as to the concept *G*'

is also true. Hence, there is complete agreement here.

To be sure, one does not say that one number encompasses more than another in the same sense that the extension of one concept encompasses more than does another; however, so is it impossible that

> the extension of the concept "equinumerous with the concept *F*"

should encompass more than

> the extension of the concept "equinumerous with the concept *G*"

Rather, if all concepts which are equinumerous with *G* are also equinumerous with *F*, then conversely, all concepts which are equinumerous with *F* are also equinumerous with *G*. This term 'more encompassing' should not, of course, be confused with the term 'greater', which occurs among numbers.

Certainly, it is also imaginable that the extension of the concept "equinumerous with the concept *F*" might encompass more or less than the extension of another concept; the latter, then, could not be a number according to our definition. Furthermore, it is not usual to call a number more or less encompassing than the extension of a concept. Nonetheless, there is nothing in the way of so speaking should the occasion arise.

Completion and confirmation of our definition

70. Definitions are confirmed by their fruitfulness. Those definitions which could just as easily be left out without invalidating proofs should be discarded as wholly worthless.

Let us see, then, whether some of the familiar properties of numbers can be derived from our definition of the number which applies to the concept *F*. We will be satisfied here by the most simple properties.

In order to do this, it is necessary to specify somewhat more exactly the meaning of equinumerosity. We defined it in terms of one-to-one correlation; just how I want to understand this expression must now be explained, since one might easily suspect a connection with intuition.

Let us consider the following example: If a waiter wants to be sure that he is placing just as many knives as plates on the table, he need count neither of them if he places a knife immediately to the right of each plate so that each knife on the table is located to the immediate right of a plate. The plates and knives are thus correlated in one-to-one fashion with one another, in this case through the same positional relationship. If, in the sentence

> 'α lies immediately to the right of *A*'

we imagine all sorts of objects substituted for α and *A*, then the part of the content which remains unchanged through all this forms the essence of the relation. Let us generalize this:

When, from a thought content which concerns an object *a* and an object *b*, we remove *a* and *b*, we retain the concept of a relation, which, accordingly, requires supplementation in two places. If, in the statement

> 'the earth has more mass than the moon',

we remove "the earth," then we obtain the concept "having more mass than the moon." If, on the other hand, we remove the object, "the moon," we gain the concept "having less mass than the earth." Removing both at once leaves a relational concept, which has in itself no more meaning than a simple concept, and which must be supplemented to become a thought content. But this supplementation can come about in various ways: instead of the earth and moon, I can take, e.g., the sun and earth, thus also effecting a removal of the earth and moon [and disclosing the relational nature of the concept].

The individual pairs of associated objects are related – one might say as subjects – to the relational concept in a manner similar to that of the individual object and the concept under which it falls. The subject here is a composite. At times, when the relation is a reversible one [symmetric in two argument places], this is also expressed linguistically, as in the sentence 'Peleus and Thetis were the parents of Achilles'.[10]

On the other hand, it would not be possible to reformulate the statement 'the earth is greater than the moon' so as to make 'the earth and the moon' appear as a compound subject, because the 'and' always indicates a certain equality of rank. This, however, does not affect the matter at hand.

The concept of relation, like the simple concept, belongs, then, to pure

[10]Do not confuse this with the case where the 'and' only seemingly connects the subjects, but in reality, however, connects two sentences.

logic. The particular content of the relation does not concern us here, but only its logical form. And [the truth of] whatever can be asserted about this form is analytic and is known *a priori*. This holds for the relational concepts as well as for the others.

Just as

'*a* falls under the concept *F*'

is the general form of a thought content concerning the object *a*, so can

'*a* stands in the relation ϕ to *b*'

be taken as the general form of a thought content concerning objects *a* and *b*.

71. Now if each object which falls under the concept *F* stands in the relation ϕ to an object falling under the concept *G*, and if, for each object which falls under *G*, there is an object falling under *F* which stands in the relation ϕ to it, then the objects falling under *F* and *G* are correlated with one another by means of the relation ϕ.

We may still ask what the expression

'each object which falls under *F* stands in the relation ϕ to an object falling under *G*'

means, if no object at all falls under *F*. By this I mean that the two statements

'*a* falls under *F*'

and

'*a* does not stand in the relation ϕ to any object falling under *G*'

cannot stand together, no matter what *a* denotes, so that either the first or the second or both are false. From this it follows that if there is no object falling under *F*, then "each object which falls under *F* stands in the relation ϕ to an object falling under *G*," because the first statement

'*a* falls under *F*'

is always to be denied, no matter what *a* might be.

Thus

'for each object which falls under *G*, there is an object falling under *F* which stands in the relation ϕ to it'

means that the two statements

'*a* falls under *G*'

and

'no object falling under *F* stands in the relation ϕ to *a*'

cannot stand together, whatever *a* may be.

72. We have now seen when the objects falling under the concepts *F* and *G* are correlated with one another by means of the relation ϕ. This correlation is here supposed to be one-to-one. By that I mean that the following two statements must hold:

1. If *d* stands in the relation ϕ to *a*, and if *d* stands in the relation ϕ to *e*, then, no matter what *d*, *a*, and *e* may be, *a* is always the same as *e*.
2. If *d* stands in the relation ϕ to *a*, and if *b* stands in the relation ϕ to *a*, then, whatever *d*, *b*, and *a* may be, *d* is always the same as *b*.

By these statements we have reduced one-to-one correlations to purely logical terms and can now offer the following definition:

the expression

'the concept *F* is equinumerous with the concept *G*'

is to be synonymous with the expression

'there is a relation ϕ which correlates in one-to-one fashion the objects falling under *F* with the objects falling under *G*'.

I [now] repeat [our original definition]:

the number which applies to the concept *F* is the extension of the concept "equinumerous with the concept *F*,"

and add to it:

the expression:

'*n* is a number'

is to be synonymous with the expression

'there is a concept to which the number *n* applies'.

Thus the concept of number is defined, apparently by means of itself, nevertheless without fallacy, because 'the number which applies to the concept *F*' has already been defined.

73. We want to show next, then, that the number which applies to the concept F is equal to the number which applies to the concept G, if the concept F is equinumerous with the concept G. This sounds like a tautology, but it is not, since the meaning of the word 'equinumerous' does not follow from its (linguistic) composition, but rather from the foregoing definition.

According to our definition, we must show that the extension of the concept "equinumerous with the concept F" is the same as that of the concept "equinumerous with the concept of G," if the concept F is equinumerous with the concept G. In other words, it must be shown that, under this hypothesis, the following statements always hold:

'if the concept H is equinumerous with the concept F, then it is also equinumerous with the concept G';

and

'if the concept H is equinumerous with the concept G, then it is also equinumerous with the concept F'.

The upshot of the first statement is that there is a relation which correlates in one-to-one fashion the objects falling under the concept H with those falling under the concept G, if there is a relation ϕ which correlates one-to-one the objects falling under the concept F with those falling under the concept G, and if there is a relation ψ which correlates one-to-one the objects falling under the concept H with those falling under the concept F. The following arrangement of the letters will make this easier to see

$$H\psi F\phi G.$$

Such a relation can in fact be given: it is [that] part of the thought content:

"there is an object to which c stands in the relation ψ and which stands in the relation ϕ to b"

[which remains] if we remove from it c and b (considering them as the things related). It can be shown that this relation is one-to-one and that it correlates the objects falling under the concept H with those falling under the concept G.

In a similar manner, the other theorem can also be proved.[11] Hopefully, these outlines will suffice to demonstrate that we need not borrow

[11]Similarly for its converse: If the number which applies to the concept F is the same as that which applies to the concept G, then the concept F is equinumerous with the concept G.

here any evidence from intuition, and that something may be done with our definitions.

74. We can now go on to the definitions of the individual numbers.

Because nothing falls under the concept "unequal to itself," I define:

0 is the number which applies to the concept "unequal to itself."

Perhaps someone will take exception to my speaking about a concept here. He will perhaps object that a contradiction is contained therein and will recall the old stand-bys, wooden iron and the square circle. To my mind, these are not at all as bad as they are made out to be. Of course, they are not exactly useful, but they can't do any harm, either, as long as one doesn't require that something fall under them; and *that* one does not yet do through the mere usage of the concepts. That a concept contains a contradiction is not always obvious without some examination; but to do that, one must have [the concept] and treat it logically just like any other. All that can be demanded of a concept from the point of view of logic and for rigor in proof procedure is its precise delineation; that, for each object, it be determined whether or not it falls under the concept. This requirement is fully satisfied, then, by concepts containing a contradiction, such as "unequal to itself," for it is known of every object that it does not fall under such a concept.[12]

I use the word 'concept' in such a way that

'a falls under the concept F'

is the general form of a thought content, which concerns an object a and which remains decidable, whatever one may put for a. And in this sense,

'a falls under the concept "unequal to itself"'

is synonymous with

'a is unequal to itself'

or

'a is unequal to a'.

In defining 0, I could have taken any other concept under which nothing

[12]Completely different from this is the definition of an object in terms of a concept under which it falls. The expression 'the greatest proper fraction' has, for example, no content, because the definite article carries with it the requirement that it refer to a definite object. On the other hand, the concept, "fraction which is less than 1 and has the property that no fraction which is less than 1 exceeds it in magnitude," is wholly unobjectionable. In fact, in order to prove that there is no such fraction, one even needs this concept, even though it contains a contradiction.

falls. It was up to me, however, to choose one of which this could be purely logically proved, and for this purpose "unequal to itself" presented itself most comfortably, whereby I let the previously presented definition of Leibniz hold, which is also purely logical.

75. We must now be able to prove, by means of what has already been said, that every concept under which nothing falls is equinumerous with any other concept under which nothing falls, and only with such a concept; from which it follows that 0 is the number which applies to such a concept and that no object falls under a concept if the number which applies to that concept is 0.

If we assume that no object falls either under the concept *F* or under the concept *G*, then, in order to prove that they are equinumerous, we need a relation ϕ about which the following statements hold:

> 'each object which falls under *F* stands in the relation ϕ to an object which falls under *G*; for each object which falls under *G* there is one falling under *F* which stands in the relation ϕ to it'.

According to what was said earlier about the meaning of these expressions, every relation fulfills these conditions under our hypotheses; hence also equality, which is, moreover, one-to-one. For, both the foregoing statements required of it hold.

If, on the other hand, an object falls under *G*, e.g., *a*, whereas none falls under *F*, then the two statements

> '*a* falls under *G*'

and

> 'no object falling under *F* stands in the relation ϕ to *a*'

hold for every relation ϕ; for, the first holds true according to the first assumption, and the second, according to the second. That is, if there is no object falling under *F*, then there is also none which would stand in any sort of relation to *a*. There is, therefore, no relation which would, according to our definition, correlate the objects falling under *F* with those falling under *G*; accordingly, the concepts *F* and *G* are not equinumerous.

76. I want now to define the relation in which any two adjoining members of the series of natural numbers stand to one another. The statement

> 'there is a concept *F* and an object *x* falling under it such that the number which applies to the concept *F* is *n*, and that the number which applies to the concept "falling under *F* but not identical with *x*" is *m*',

is to be synonymous with

> '*n* immediately follows *m* in the series of natural numbers'.

I am avoiding the expression '*n* is *the* number immediately following *m*', because two theorems would first have to be proved in order to justify the use of the definite article.[13] For the same reason, I am not yet saying here '$n=m+1$'; for, by means of the equals sign, $(m+1)$ is also designated as an object.

77. Now in order to arrive at the number 1, we must first show, that there is something which immediately follows 0 in the series of natural numbers.

Let us consider the concept – or, if you prefer – the predicate 'equal to 0'. 0 falls under this. On the other hand, no object falls under the concept "equal to 0 but not equal to 0," so that 0 is the number which applies to this concept. We have therefore, a concept "equal to 0" and an object 0 falling under it, for which it holds that:

> the number which applies to the concept "equal to 0" is equal to the number which applies to the concept "equal to 0";
>
> the number which applies to the concept "equal to 0 but not equal to 0" is 0.

Therefore, according to our definition, the number which applies to the concept "equal to 0" follows immediately after 0 in the series of natural numbers.

If we define, then,

> 1 is the number which applies to the concept "equal to 0,"

then we can express the last statement so:

> 1 immediately follows 0 in the series of natural numbers.

Perhaps it is not superfluous to note that the definition of 1 does not presuppose any observed fact[14] for its objective legitimacy, for one can easily be confused by the fact that certain subjective conditions must be fulfilled in order to enable us to give the definition, and that sense impressions cause us to do so (cf. Erdmann 1877: 164). This can, nevertheless, be the case without the derived theorems ceasing to be *a priori*. To such conditions belongs the requirement, for example, that blood

[13]See footnote 12.
[14]A proposition that is not general.

flow through the brain in sufficient quantity and of the right concentration – at least as far as we know; however, the truth of our last proposition is independent of that; it continues to hold even if this flow no longer takes place. And even if all reasonable creatures should at some time simultaneously slip into hibernation, the truth of the statement would not, as it were, be suspended for the duration of this sleep, but would remain undisturbed. The truth of a statement is not its being thought.

78. I list here several theorems to be proved by means of our definitions. The reader will easily see how this may be done.

I. If a immediately follows 0 in the series of natural numbers, then $a=1$.

II. If 1 is the number which applies to a concept, then there is an object which falls under that concept.

III. If 1 is the number which applies to a concept F; if the object x falls under the concept F, and if y falls under the concept F, then $x=y$; i.e., x is the same as y.

IV. If an object falls under a concept F and if, from the fact that x falls under the concept F and that y falls under the concept F, it may always be inferred that $x=y$, then 1 is the number which applies to the concept F.

V. The relation that m bears to n, if and only if

"n immediately follows m in the series of natural numbers",

is a one-one relation.

Thus far it has not yet been said that for every number there is another which immediately follows it or is immediately followed by it in the series of natural numbers.

VI. Every number except 0 immediately follows another number in the series of natural numbers.

79. Now in order to be able to prove that every number (n) in the series of natural numbers is immediately followed by a number, one must come up with a concept to which this latter number applies. We choose for this:

"belonging to the series of natural numbers ending with n,"

but we must first define it.

To begin with I shall repeat, in somewhat different words, the definition I gave in my *Begriffsschrift* of following in a series:

The statement

'if every object to which x stands in the relation ϕ falls under the concept F, and if, from the fact that d falls under the concept F, it always follows, no matter what d may be, that every object to which d stands in the relation ϕ falls under the concept F, then y falls under the concept F, no matter what concept F might be',

is to be synonymous with

'y follows x in the ϕ-series'

and with

'x precedes y in the ϕ-series'.

80. Several remarks concerning this definition will not be superfluous here. Since the relation ϕ is left indeterminate, the series is not necessarily to be thought of in the form of a spatial or temporal arrangement, although these cases are not excluded.

Some other definition might be considered more natural, e.g., if, in proceeding from x, we always turn our attention from one object to another, to which it stands in the relation ϕ, and if, in this way, we can finally reach y, then we say that y follows x in the ϕ-series.

This is a way of looking at the matter, not a definition. Whether we reach y in the wanderings of our attention can depend on many subjective incidental circumstances; e.g., on the time we have available or on our knowledge of the things. Whether y follows x in the ϕ-series has, in general, nothing at all to do with our attention and the conditions of its progress, but rather it is a matter of objective fact: just as a green leaf reflects certain light rays whether or not they should meet my eye and summon up a sensation; just as a grain of salt is soluble in water whether or not I put it in water and observe the process; and just as it remains soluble even if it is not possible for me to experiment on it.

By means of my definition, the matter is elevated from the realm of the subjectively possible to that of the objectively definite. Indeed, the fact that from certain statements another statement follows is something objective, something independent of whatever laws may govern the wanderings of our attention; and it makes no difference whether we really make the inference or not. Here we have a criterion which decides the question, wherever it can be asked, even though we might be hindered by external difficulties from judging in individual cases whether it is applicable. That makes no difference to the issue itself.

We need not always run through all the intermediate members, from the initial member up to an object, in order to be sure that the latter follows the former. If, e.g., it is given that, in the ϕ-series, b follows a and c follows b, then we can conclude on the basis of our definition that c follows a, without even knowing the intermediate members.

Only by means of this definition of following in a series does it become possible to reduce the rule of inference from n to $(n+1)$, which apparently is peculiar to mathematics, to general logical laws.

81. Now if we have as our relation ϕ the one in which m is related to n by the statement

> 'n immediately follows m in the series of natural numbers',

then we say instead of 'ϕ-series', 'series of natural numbers'.

I define further:

the statement

> 'y follows x in the ϕ-series or y is the same as x',

is to be synonymous with

> 'y belongs to the ϕ-series starting with x'

and with

> 'x belongs to the ϕ-series ending with y'.

According to this, a belongs to the series of natural numbers ending with n if n either follows a in the series of natural numbers or is equal to a.[15]

82. We must now show that, under a condition still to be stated, the number which applies to the concept

> "belonging to the series of natural numbers ending with n"

immediately follows n in the series of natural numbers. Having this result, we will have proved that there is a number which immediately follows n in the series of natural numbers; i.e., that there is no last member of this series. Obviously, this statement cannot be established empirically or by means of induction.

It would take us too far afield to give the proof itself. We can only give a brief sketch of it here. We must prove:

1. If a immediately follows d in the series of natural numbers, and if the number which applies to the concept

   > "belonging to the series of natural numbers ending with d"

[15] If n is not a number, then only n itself belongs to the series of natural numbers ending with n. One should not object to this expression.

   immediately follows d in the series of natural numbers, then the number which applies to the concept

   > "belonging to the series of natural numbers ending with a"

   immediately follows a in the series of natural numbers.
2. We must prove that what has been asserted about d and a in the foregoing statements holds for 0, and then show that it also holds for n, if n belongs to the series of natural numbers beginning with 0. This will result from an application of my definition of

   > 'y follows x in the series of natural numbers',

   taking as the concept F the relation asserted above to hold between d and a, and substituting 0 and n for d and a.

83. In order to prove Theorem 1 of the last paragraph, we must show that a is the number which applies to the concept "belonging to the series of natural numbers ending with a, but not equal to a." And to this end, we must prove that this concept has the same extension as the concept "belonging to the series of natural numbers ending with d." For this, we need the theorem that no object which belongs to the series of natural numbers beginning with 0 can follow itself in the series of natural numbers. The latter must likewise be proved by means of our definition of following in a series, as it is outlined above.[16]

For this reason, we must add the condition that n belong to the series of natural numbers beginning with 0 to the statement that the number which applies to the concept

> "belonging to the series of natural numbers ending with n,"

immediately follows n in the series of natural numbers. There is a shorter way of putting this, which I shall now define:

the statement

> 'n belongs to the series of natural numbers beginning with 0'

is to be synonymous with

> 'n is a finite number'.

We can now express the last theorem thus: no finite number follows itself in the series of natural numbers.

[16] E. Schröder (1873: 63) seems to look upon this theorem as the consequence of an ambiguous terminology. The difficulty which infects his whole presentation of the matter emerges here too; i.e., it is never quite clear whether the number is a symbol and, if so, what its meaning is, or whether it *is* this very meaning. From the fact that one sets up different symbols, so that the same one never recurs, it does not follow that these symbols mean different things.

Infinite numbers

84. In contrast to the finite numbers there are the infinite ones. The number which applies to the concept "finite number" is an infinite one. Let us denote it, say, by $\aleph_0$.[17] Were it a finite number, it could not follow itself in the series of natural numbers. One can show, however, that $\aleph_0$ does just this.

There is nothing somehow mysterious or marvellous about the infinite number $\aleph_0$ when so defined. 'The number which applies to the concept *F* is $\aleph_0$' says nothing more nor less than: there is a relation which establishes a one-to-one correlation between the objects falling under the concept *F* and the finite numbers. This has, according to our definitions, a completely clear and unambiguous sense, and that suffices to justify the use of the symbol $\aleph_0$ and to guarantee it a meaning. That we can form no mental picture of an infinite number is wholly irrelevant and would hold true of finite numbers as well. In this way, our number $\aleph_0$ is something just as determinate as any finite number: it can be recognized without a doubt as the same and differentiated from any other.

85. Recently, in a noteworthy paper (1883b), G. Cantor introduced infinite numbers. I agree with him completely in his evaluation of the view which would have only the finite numbers qualify as real. Neither these nor the fractions are sensibly perceptible and spatial, nor are the negative, irrational, and complex numbers. And if one calls real [only] that which affects the senses, or at least can have sense impressions as an immediate or distant consequence, then certainly none of these numbers is real. But we don't need such sense impressions as evidence for our theorems. A name or a symbol, which is introduced in a logically unobjectionable way, may be used by us without hesitation in our investigations, and thus our number $\aleph_0$ is just as firmly grounded as 2 or 3.

Although I believe I agree with Cantor in this matter, I do, however, deviate from him in terminology. He calls my numbers 'powers', whereas his concept[18] of number is based on ordering. To be sure, finite numbers end up being independent of order; however, this does not hold for infinite numbers. Now the linguistic usage of the word 'number' and of the question 'how many?' contains no indication of a definite order. Cantor's number answers rather the question: 'the last member is the how-manyth member of the sequence?' Therefore my terminology seems to me to

[17] [Frege used '∞_1', but we adopt the aleph notation as being more in keeping with current practice. – Tr.]

[18] This expression may appear to contradict [my earlier remarks emphasizing] the objectivity of concepts; however, only the *terminology* is subjective here.

agree better with linguistic usage. If one extends the meaning of a word, then one must take care that as many general statements as possible retain their validity, and particularly statements as basic as, for instance, [the one asserting] for numbers their independence of the sequence. We have needed no extension at all, because our concept of number immediately embraces infinite numbers as well.

86. In order to obtain his infinite numbers, Cantor introduces the relational concept of following in a sequence, which differs from my "following in a series." According to him, for instance, a sequence would result if one were so to order the finite positive whole numbers that the odd numbers followed one another in their own natural order, and similarly the even numbers in theirs, and it were further stipulated that all the even numbers should come after all the odd numbers. In this sequence, e.g., 0 would follow 13. There would, however, be no number immediately preceding 0. Now this case cannot occur within my definition of following in a series. It may be strictly proved, without using intuition, that, if y follows x in the ϕ-series, there is an object which immediately precedes y in this series. It seems to me, then, that exact definitions of following in a sequence and of number [in Cantor's sense] are still lacking. Thus Cantor bases himself on a somewhat mysterious "inner intuition" where a proof from definitions should be striven for and would probably be found. For I think I can foresee how those concepts could be defined. In any case, I in no way wish these comments to be taken as an attack on the justifiability or fruitfulness of these concepts. On the contrary, I welcome these investigations as an extension of the science, especially because they strike a purely arithmetic path to higher infinite numbers (powers).

Conclusion

87. I hope in this monograph to have made it probable that arithmetic laws are analytic judgments, and therefore *a priori*. According to this, arithmetic would be only a further developed logic, every arithmetic theorem a logical law, albeit a derived one. The applications of arithmetic to the explanation of natural phenomena would be logical processing of observed facts;[19] computation would be inference. Numerical laws will not need, as Baumann (1868–9, 2: 670) contends, a practical confirmation in order to be applicable in the external world; for, in the external world, the totality of space and its contents, there are no concepts, no properties of concepts, no numbers. Therefore, the numerical laws are

[19] Observation itself already includes a logical activity.

really not applicable to the external world: they are not laws of nature. They are, however, applicable to judgments, which are true of things in the external world: they are laws of the laws of nature. They assert connections not between natural phenomena, but rather between judgments; and it is to the latter that the laws of nature belong.

88. Kant (1867–8, 3: 39ff) evidently underestimated the value of analytic judgments – probably as the result of having too narrow a definition of the concept – although he apparently also had in mind the broader concept used here.[20] Taking his definition as a basis, the division of judgments into the analytic and the synthetic is not exhaustive. He is thinking of universal affirmative judgments. In such cases, one can speak of a concept of the subject and inquire whether the concept of the predicate – as would result from *his* definition – is contained in it. How can we do this, however, when the subject is a single object? Or when the judgment is existential? In such cases there can be, in Kant's sense, no talk of a concept of the subject. Kant seems to have thought of the concept as determined by subordinate characteristics; that, however, is one of the least fruitful notions of concept. If one surveys the foregoing definitions, one will hardly find one of this kind. The same is true of the really fruitful definitions in mathematics, e.g., of the continuity of a function. There we don't have a series of subordinate characteristics but rather a more intimate, I should say more organic, connection between the [elements of the] definitions. The difference can be illustrated by means of a geometrical analogy. If the concepts (or their extensions) are represented by regions of a plane, then the concept defined by means of subordinate characteristics corresponds to the region which is the overlap of all the individual regions corresponding to these characteristics; it is enclosed by parts of their boundaries. Pictorially speaking, in such a definition, we delimit a region by using in a new way lines already given. In doing this, however, nothing essentially new comes out. The more fruitful definitions draw border lines which had not previously been given. What can be inferred from them cannot be seen in advance; one does not simply withdraw again from the box what one has put into it. These inferences expand our knowledge and one should, therefore, following Kant, consider them synthetic. Nevertheless, they can be proved purely logically and hence are analytic. They are in fact contained in the definitions, but like the plant in the seed, not like the rafter in the house. One often needs several definitions to prove a theorem, which consequently is contained in no single definition, but nevertheless follows in a purely logical way from all of them together.

[20][Kant] says that a synthetic statement can be understood according to the Theorem of Contradiction only if another synthetic statement is presupposed (1867–8, 3: 43).

89. I must also contradict the generality of Kant's assertion (1867–8, 3: 82) that without sensible perception no object would be given us. Zero and 1 are objects that cannot be given us sensibly. And those who hold the smaller numbers to be intuitive will surely have to concede that none of the numbers greater than $1000^{1000^{1000}}$ can be given them intuitively, and that we nevertheless know a good deal about them. Perhaps Kant was using the word 'object' in a somewhat different sense; but then zero, 1, and our $\aleph_0$ disappear completely from his considerations; for, they are not concepts either, and Kant demands even of concepts that their objects be appended to them in intuition.

In order not to open myself to the criticism of carrying on a picayune search for faults in the work of a genius whom we look up to only with thankful awe, I believe I should also emphasize our areas of agreement, which are far more extensive than those of our disagreement. To touch on only the immediate points, I see a great service in Kant's having distinguished between synthetic and analytic judgments. In terming geometric truths synthetic and *a priori,* he uncovered their true essence. And this is still worth repeating today, because it is still often not recognized. If Kant erred with respect to arithmetic, this does not detract essentially, I think, from his merit. It was important for him that there should be synthetic judgments *a priori;* whether they occur only in geometry or also in arithmetic is of little importance.

90. I do not claim to have made the analytic nature of arithmetic theorems more than probable, because one can always still doubt whether their proof can be carried out completely from purely logical laws, whether evidence of another sort has not crept in unnoticed somewhere. This doubt is also not entirely relieved by the outlines which I have given of the proofs of a few theorems; it can only be alleviated by an airtight chain of reasoning, such that no step is made which is not in conformity with one of a few rules of inference recognized as purely logical. Thus until now, hardly a single [real] proof has ever been offered, because the mathematician is satisfied if every transition to a new judgment appears to him to be correct, without asking whether this appearance is logical or intuitive. A step in such a proof is often quite complex and involves several simple inferences, in addition to which intuitive considerations can creep in. One proceeds in jumps, and from this there arises the impression of an over-rich variety of rules of inference used in mathematics. For, the greater the jumps, the more complex are the combinations of simple inferences and intuitive axioms which they can represent. Nevertheless, such a transition often occurs to us directly, without our being conscious of the intermediate steps, and since it does not present itself as one of the recognized logical rules of inference, we are immediately ready to consider this manifest transition as an intuitive one and the inferred truth as a synthetic one, even when the range of its validity extends far beyond intuition.

Proceeding in this way, it is not possible clearly to separate the synthetic, based on intuition, from the analytic. Nor will it be possible to compile with completeness and certainty the axioms of intuition needed to make every mathematical proof capable of proceeding from these axioms alone, according to logical laws.

91. The requirement of avoiding all jumps in a proof must, therefore, be imposed. That it is so difficult to satisfy is owing to the tediousness of a step-by-step procedure. Every proof, which is even slightly involved, threatens to become enormously long. In addition to this, the superfluity of logical forms expressed in language makes it difficult to extract a group of rules of inference sufficient for all cases and yet easy to survey.

In order to minimize the effects of these drawbacks, I have devised my concept writing. It strives for greater brevity and comprehensibility of expression and is manipulated in a few standard ways, as in a computation, so that no transition is permitted which does not conform to rules set up once for all.[21] No assumption can then slip in unnoticed. I have thus proved a theorem,[22] borrowing no axioms from intuition, which one would consider at first glance to be synthetic and which I shall state here as follows:

If the relation of each member of a series to its successor is one-to-one, and if *m* and *y* follow *x* in this series, then *y* precedes *m* in this series, or coincides with it, or follows *m*.

From this proof, one can see that theorems which expand our knowledge can contain analytic judgments.[23]

[21] It is, however, supposed to be able to express not only the logical form of a statement, as does the Boolean notation, but also its content.

[22] *Begriffsschrift*, 1879, p. 86, Formula 133.

[23] This proof will be found to be still much too lengthy, a disadvantage which may seem to more than balance out the almost unconditional guarantee against a mistake or a loophole. My purpose at that time was to reduce everything to the smallest possible number of the simplest possible logical laws. As a result of this, I applied only one rule of inference. I pointed out even then, in the foreword (p. vii) that, for further application, it would be recommended to admit more rules of inference. This can be done without impairing the validity of the chain of reasoning, and an important abbreviation could thereby be achieved.

[Recapitulation]

106. Let us cast a quick glance backward on the course of our investigation. After determining that a number was not a collection of things nor a property of such a collection, nor, furthermore, the subjective product of mental processes, we decided that a statement of number asserts something objective about a concept. We defined first the individual numbers 0, 1 etc., and then following in the number series. Our first attempt failed, because in it we stated the meaning of only whole assertions about concepts, and not of 0 and 1 separately, although these entered into those assertions. As a result of this, we could not prove the equality of numbers. It was shown that the numbers with which arithmetic concerns itself must be understood not as dependent attributes, but rather substantivally.[24] Thus numbers appeared to us as recognizable objects, although not physical ones nor even merely spatial ones, nor ones which we could picture in imagination. We then established the basic theorem: that the meaning of a word is not to be defined separately, but rather in the context of a statement; only by following this theorem can we, I think, avoid the physical interpretation of number, without slipping into psychological interpretation. There is only one type of statement which must have a sense for every object; that is the recognition sentences, called equations in the case of numbers. We saw that statements of number are also to be interpreted as equations. It became a question, then, of determining the sense of a numerical equation and of expressing this sense without making use of the number-words or the word 'number'. The possibility of establishing a one-to-one correspondence between the objects falling under a concept *F* and those falling under a concept *G* was found to be the content of a recognition judgment about numbers. Our definition, therefore, had to posit that possibility as synonymous with a numerical equation. We recalled similar instances: the definition of direction from parallelism, of shape from similarity, etc.

107. The question then arose: when are we justified in interpreting a content to be that of a recognition judgment? For this, the condition must be fulfilled that in every judgment the left side of the tentatively assumed equation can be replaced by the right, without altering the truth of the judgment. Now, at first and without resorting to further definitions, no further assertion about the left or right side of such an equation is known to us beyond the assertion of their equality. Substitutivity had therefore to be proved only for equations.

[24] The difference corresponds to that between 'blue' and 'the color of the sky'.

A doubt still remained, however. A recognition statement must always have a sense. If we interpreted the possibility of correlating in one-to-one fashion the objects falling under the concept *F* with those falling under the concept *G* as an equation, by saying for it: 'the number which applies to the concept *F* is equal to the number which applies to the concept *G*' and thereby introducing the expression 'the number which applies is the concept *F*', then we have a sense for the equation only if both sides have the form just mentioned. We would not be able to judge according to such a definition whether an equation only one side of which had this form was true or false. That caused us to make the following definition:

> The number which applies to the concept *F* is the extension of the concept "concept equinumerous with the concept *F*,"

by which we called a concept *F* equinumerous with a concept *G*, if there exists the possibility of correlating them one-to-one.

In doing this, we presuppose that the sense of the expression 'extension of a concept' is familiar. This method of overcoming the difficulty will probably not be everywhere applauded, and some will prefer to set aside this doubt in another way. I, too, place no decisive weight on the introduction of the extension of a concept.

108. We still had to define one-to-one correspondences; we reduced them to purely logical terms. After we had outlined the proof of the theorem that the number which applies to the concept *F* is equal to that which applies to the concept *G*, if the concept *F* is equinumerous with the concept *G*, we defined 0, the expression '*n* immediately follows *m* in the series of natural numbers', and the number 1, and we showed that 1 immediately follows 0 in the series of natural numbers. We presented a few theorems which could be easily proved at this point and then went somewhat more deeply into the following, which demonstrates the infinity of the number series:

> Every number in the series of natural numbers is followed by a number.

We were thereby led to the concept "belonging to the series of natural numbers ending with *n*," of which we wanted to show that the number applying to it immediately follows *n* in the series of natural numbers. We defined it at first by means of an object *y* following an object *x* in a general ϕ-series. The sense of this expression was also reduced to purely logical terms. And thereby we succeeded in proving that the rule of inference from *n* to $(n+1)$, which is usually considered a peculiarly mathematical one, is based on the general logical rules of inference.

For the proof of the infinity of the number series, we needed the theorem that no finite number follows itself in the series of natural numbers. We thus arrived at the concepts of finite and infinite numbers. We showed that the latter is basically no less justified logically than is the former. For the purposes of comparison, we drew upon Cantor's infinite numbers and his "following in a sequence," where we pointed out the difference in terminology.

109. We thus rendered the analytic and *a priori* character of arithmetic truths highly probable, arriving at an improvement on Kant's point of view. We saw further what was still lacking in order to elevate that probability to certainty and we indicated the path that must lead to this.